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SUBMANIFOLDS OF CODIMENSION 3 OF A KAEHLERIAN MANIFOLD (II)

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0. Introduction

As is well known, a submanifold of codimension 3 of an almost Hermitian manifold admits the so-called $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced from the almost Hermitian structure of the ambient manifold ([5], [11]).

Recently Yano and one of the present authors ([11]) have studied the condition that the induced $(f, g, u, v, w, \lambda, \mu, \nu)$-structure defines an almost contact metric structure, and properties of a pseudo-umbilical submanifold of codimension 3 in an even-dimensional Euclidean space satisfying the assumption above.

On the other hand, the present authors ([2]) have generalized the facts by studing a submanifold of codimension 3 of a complex space form.

The purpose of the present paper is to research the intrinsic character of a submanifold of codimension 3 with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$ of a real $2m$-dimensional complex projective space CP^m by method of Riemannian fibre bundle under the condition that the second fundamental tensors commute with f each other. As a special case of a complex space form, we also study a submanifold of codimension 3 in an even-dimensional Euclidean space satisfying the above assumptions.

In §1, we recall the fundamental relations and structure equations of a submanifold M of codimension 3 admitting an almost contact metric structure in a Kaehlerian manifold. In §2, under the conditions that the second fundamental tensors commute with the structure tensor f, we find the covariant derivative of a second fundamental tensor of the submanifold M of codimension 3 in a complex projective space. In §3, we characterize a submanifold of codimension 3 of an even-dimensional Euclidean space satisfying the assumptions as stated in §2, In the last §4, using the theory of Riemannian submersion and immersion, we prove that the submanifold M of codimension 3 of a complex projective space CP^m becomes a model space $M_{p,q}{}^C(a, b)=\tilde{\pi}(S^{2p+1}(a)\times S^{2q+1}(b))$, $S^p(a)$ being a p-dimensional

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sphere of radius $a>0$, where $\tilde{\pi}$ is a natural projection of a principal circle bundle S^{2m+1} onto CP^m defined by the Hopf-fibration, (p, q) is some portion of $m-2$ and $a^2+b^2=1$.

We introduce the following theorem for later use.

THEOREM A. ([11]). *Let M^{2n+1} be a pseudo-umbilical submanifold of an even-dimensional Euclidean space E^{2n+4} with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. Then M^{2n+1} is an intersection of a complex cone with generator as a distinguished normal and a $(2n+3)$-dimensional sphere.*

1. Submanifolds of codimension 3 in an almost Hermitian manifold.

Let M^{2n+4} be a $(2n+4)$-dimensional almost Hermitian manifold covered by a system of coordinate neighborhoods $\{\tilde{U}; x^A\}$, denoted by g_{CB} components of the Hermitian metric tensor and by $F_B{}^A$ those of the almost complex structure tensor of M^{2n+4}, where here and in the sequel the indices $A, B, C, \cdots$ run over the range $\{1, 2, \cdots, 2n+4\}$. Then we have

(1.1) $$F_C{}^B F_B{}^A = -\delta_C{}^A, \quad g_{ED} F_C{}^E F_B{}^D = g_{CB},$$

$\delta_C{}^A$ being the Kronecker delta.

Let M^{2n+1} be a $(2n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and immersed isometrically in M^{2n+1} by the immersion $i: M^{2n+1} \longrightarrow M^{2n+4}$, where here and in the sequel the indices $h, j, i, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$. We identify $i(M^{2n+1})$ with M^{2n+1} itself and represent the immersion by

(1.2) $$x^A = x^A(x^h).$$

We now put $B_i{}^A = \partial_i x^A$, $(\partial_i = \partial/\partial x^i)$. Then $B_i{}^A$ are $2n+1$ linearly independent vectors of M^{2n+4} tangent to M^{2n+1}. And denote by C^A, D^A and E^A three mutually orthogonal unit normals to M^{2n+1}. Then denoting by g_{ji} components of the induced metric tensor of M^{2n+1}, we have

(1.3) $$g_{ji} = g_{CB} B_j{}^C B_i{}^B$$

since the immersion is isometric.

The transforms of $B_i{}^A$, C^A, D^A and E^A by $F_B{}^A$ can be respectively expressed as

(1.4) $$F_B{}^A B_i{}^B = F_i{}^h B_h{}^A + u_i C^A + v_i D^A + w_i E^A,$$

(1.5) $$F_B{}^A C^B = -u^h B_h{}^A - \nu D^A + \mu E^A,$$

(1.6) $$F_B{}^A D^B = -v^h B_h{}^A + \nu C^A - \lambda E^A,$$

(1.7) $$F_B{}^A E^B = -w^h B_h{}^A - \mu C^A + \lambda D^A,$$

where $f_i{}^h$ are components of a tensor field of type $(1, 1)$, u_i, v_i and w_i those of 1-forms, λ, μ, ν functions in M^{2n+1}, and u^h, v^h, w^h vector fields associated with u_i, v_i and w_i respectively.

Applying the operator F to both sides of (1.4)–(1.7), using (1.1) and those equations and comparing tangental parts and normal parts of both sides, we find

(1.8) $$f_j{}^t f_t{}^h = -\delta_j^h + u_j u^h + v_j v^h + w_j w^h,$$

(1.9) $$\begin{cases} f_t{}^h u^t = \nu v^h - \mu w^h, \\ f_t{}^h v^t = -\nu u^h + \lambda w^h, \\ f_t{}^h w^t = \mu u^h - \lambda v^h, \end{cases}$$

(1.10) $$\begin{cases} u_t u^t = 1 - \mu^2 - \nu^2, & u_t v^t = \lambda\mu, \\ v_t v^t = 1 - \nu^2 - \lambda^2, & v_t w^t = \mu\nu, \\ w_t w^t = 1 - \lambda^2 - \mu^2, & u_t w^t = \lambda\nu. \end{cases}$$

Also, from (1.1), (1.3) and (1.4), we find

(1.11) $$g_{ts} f_j{}^t f_i{}^s = g_{ji} - u_j u_i - v_j v_i - w_j w_i.$$

If we put $f_{ji} = f_j{}^t g_{ti}$, then we easily see that $f_{ji} = -f_{ij}$. Thus (1.8)–(1.11) show that the aggregate $(f_j{}^h, g_{ji}, u_i, v_i, w_i, \lambda, \mu, \nu)$ defines the so-called $(f, g, u, v, w, \lambda, \mu, \nu)$-structure on M^{2n+1} ([5], [11]).

We find from (1.9)

(1.12) $$f_t{}^h p^t = 0,$$

where we have put

(1.13) $$p^h = \lambda u^h + \mu v^h + \nu w^h.$$

From this and (1.10), we have

(1.14) $$u_t p^t = \lambda, \quad v_t p^t = \mu, \quad w_t p^t = \nu, \quad p_t p^t = \lambda^2 + \mu^2 + \nu^2.$$

We now suppose that the aggregate $(f_i{}^h, g_{ji}, p^h)$ defines an almost contact metric structure. Then we get from the last equation of (1.14)

(1.15) $$\lambda^2 + \mu^2 + \nu^2 = 1,$$

because of $p_t p^t = 1$. Conversely, if the functions λ, μ and ν satisfy (1.15), then (1.10) reduces to

(1.16) $$\begin{aligned} &u_t u^t = \lambda^2, \quad u_t v^t = \lambda\mu, \quad u_t w^t = \lambda\nu, \\ &v_t v^t = \mu^2, \quad v_t w^t = \mu\nu, \quad w_t w^t = \nu^2. \end{aligned}$$

Computing the square of lengths of vectors $u_i - \lambda p_i$, $v_i - \mu p_i$ and $w_i - \nu p_i$, it follows that

(1.17) $$u_i = \lambda p_i, \quad v_i = \mu p_i, \quad w_i = \nu p_i,$$

where $p_i = g_{ti} p^t$ with the aid of (1.14) and (1.16).

Substituting (1.17) into (1.8) gives $f_i{}^t f_t{}^h = -\delta_i^h + p_i p^h$ because of (1.15).

Also substituting (1.17) into (1.11) and using (1.15), we find $g_{ts} f_j{}^t f_i{}^s = g_{ji} - p_j p_i$. Thus we see that the aggregate $(f_j{}^h, g_{ji}, p^h)$ defines an almost contact metric structure.

Hence we have the following

THEOREM 1.1 ([11]). *Let M^{2n+1} be a differentiable manifold with an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure. In order for the aggregate (f, g, p), p being given by* (1.13), *to define an almost contact metric structure, it is necessary and* *sufficient that* $\lambda^2+\mu^2+\nu^2=1$.

From now on, we suppose that the condition $\lambda^2+\mu^2+\nu^2=1$ is satisfied on M^{2n+1}, that is, the aggregate (f, g, p) defines an almost contact metric structure. Then we have from (1.4) and (1.17)

$$F_B{}^A B_i{}^B = f_i{}^h B_h{}^A + p_i N^A, \tag{1.18}$$

where

$$N^A = \lambda C^A + \mu D^A + \nu E^A \tag{1.19}$$

is an intrinsically defined unit normal to M^{2n+1} because C^A, D^A and E^A are mutually orthogonal unit normals to M^{2n+1} and $\lambda^2+\mu^2+\nu^2=1$.

When a submanifold of an almost Hermitian manifold satisfies the equation (1.18), N^A being a unit normal to the submanifold, we say that the submanifold is *semi-invariant* with respect to N^A ([1], [10]). We call N^A the *distinguished normal* to the semi-invariant submanifold. We take N^A as C^A. Then we have $\lambda=1$, $\mu=0$, $\nu=0$ and consequently $u^h=p^h$, $v^h=w^h=0$ because of (1.10) and (1.17).

Hence, (1.4) – (1.7) reduce respectively to

$$F_B{}^A B_i{}^B = f_i{}^h B_h{}^A + p_i C^A, \tag{1.20}$$

$$\begin{aligned} &F_B{}^A C^B = -p^h B_h{}^A, \\ &F_B{}^A D^B = -E^A \\ &F_B{}^A E^B = D^A. \end{aligned} \tag{1.21}$$

Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g_{ji}, we have equations of Gauss for M^{2n+1} of M^{2n+4}

$$\nabla_j B_i{}^A = H_{ji} C^A + K_{ji} D^A + L_{ji} E^A, \tag{1.22}$$

where H_{ji}, K_{ji}, L_{ji} are the second fundamental tensors with respect to normals C^A, D^A, E^A respectively, and those of Weingarten

$$\nabla_j C^A = -H_j{}^h B_h{}^A + l_j D^A + m_j E^A, \tag{1.23}$$

$$\nabla_j D^A = -K_j{}^h B_h{}^A - l_j C^A + n_j E^A, \tag{1.24}$$

$$\nabla_j E^A = -L_j{}^h B_h{}^A - m_j C^A - n_j D^A, \tag{1.25}$$

where $H_j{}^h = H_{jt} g^{th}$, $K_j{}^h = K_{jt} g^{th}$, $L_j{}^h = L_{jt} g^{th}$, $(g^{ji}) = (g_{ji})^{-1}$, l_j, m_j and n_j are components of the third fundamental tensors.

We now assume that the ambient manifold M^{2n+4} is a Kaehlerian manifold. Differentiating (1.20) covariantly along M^{2n+1} and making use of (1.22) and (1.23), we easily find ([11])

$$\nabla_j f_i{}^h = -H_{ji} p^h + H_j{}^h p_i, \tag{1.26}$$

$$\nabla_j p_i = -H_{jt} f_i{}^t, \tag{1.27}$$

$$K_{ji} = -L_{jt} f_i{}^t - m_j p_i, \tag{1.28}$$

$$L_{ji} = K_{jt} f_i{}^t + l_j p_i, \tag{1.29}$$

from which

(1.30) $K_{jt}p^t=-m_j,$

(1.31) $L_{jt}p^t=l_j,$

(1.32) $K=-m_tp^t,$

(1.33) $L=l_tp^t,$

where we have put $K=K_{ji}g^{ji}$, $L=L_{ji}g^{ji}$.

The equations of Gauss for M^{2n+1} in Kaehlerian manifold M^{2n+4} are given by

(1.34) $$K_{kji}{}^h=K_{DCB}{}^AB_k{}^DB_j{}^CB_i{}^BB^h{}_A+H_k{}^hH_{ji}-H_j{}^hH_{ki}+K_k{}^hK_{ji}-K_j{}^hK_{ki}+L_k{}^hL_{ji}-L_j{}^hL_{ki},$$

where $B^h{}_A=g_{AB}g^{ih}B_i{}^B$, $K_{DCB}{}^A$ and $K_{kji}{}^h$ being the Riemann-Christoffel curvature tensors of M^{2n+1} and M^{2n+4} respectively.

We now assume that the ambient manifold is a Kaehlerian manifold M^{2n+4}(c) of constant holomorphic sectional curvature c, that is, its curvature tensor has the form

(1.35) $$K_{DCB}{}^A=\frac{c}{4}(\delta_D{}^Ag_{CB}-\delta_C{}^Ag_{DB}+F_D{}^AF_{CB}-F_C{}^AF_{DB}-2F_{DC}F_B{}^A).$$

Substituting (1.35) into (1.34) and using (1.3), (1.20) and (1.21), we have

(1.36) $$K_{kji}{}^h=\frac{c}{4}(\delta_k{}^hg_{ji}-\delta_j{}^hg_{ki}+f_k{}^hf_{ji}-f_j{}^hf_{ki}-2f_{kj}f_i{}^h)+H_k{}^hH_{ji}-H_j{}^hH_{ki}+K_k{}^hK_{ji}-K_j{}^hK_{ki}+L_k{}^hL_{ji}-L_j{}^hL_{ki}.$$

Similarly we can obtain the equations of Codazzi for M^{2n+4}(c) by using (1.20) and (1.21)

(1.37) $$\nabla_kH_{ji}-\nabla_jH_{ki}-l_kK_{ji}+l_jK_{ki}-m_kL_{ji}+m_jL_{ki}=\frac{c}{4}(p_kf_{ji}-p_jf_{ki}-2p_if_{kj}),$$

(1.38) $$\nabla_kK_{ji}-\nabla_jK_{ki}+l_kH_{ji}-l_jH_{ki}-n_kL_{ji}+n_jL_{ki}=0,$$

(1.39) $$\nabla_kL_{ji}-\nabla_jL_{ki}+m_kH_{ji}-m_jH_{ki}+n_kK_{ji}-n_jK_{ki}=0.$$

Those of Ricci are given by

(1.40) $$\nabla_kl_j-\nabla_jl_k+H_k{}^tK_{jt}-H_j{}^tK_{kt}+m_kn_j-m_jn_k=0,$$

(1.41) $$\nabla_km_j-\nabla_jm_k+H_k{}^tL_{jt}-H_j{}^tL_{kt}+n_kl_j-n_jl_k=0,$$

(1.42) $$\nabla_kn_j-\nabla_jn_k+K_k{}^tL_{jt}-K_j{}^tL_{kt}+l_km_j-l_jm_k=\frac{c}{2}f_{kj}.$$

2. Submanifolds of codimension 3 in a complex projective space admiting $\lambda^2+\mu^2+\nu^2=1$.

In this section we assume that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on a submanifold M^{2n+1} $(n>1)$ of codimension 3 in a Kaehlerian manifold $M^{2n+4}(c)$ of constant holomorphic sectional curvature c satisfying $\lambda^2+\mu^2+\nu^2=1$ and consequently the aggregate (f, g, p) defines an almost contact metric structure.

We now suppose that the second fundamental tensor commute with f, that is,

$$H_j{}^t f_t{}^h - f_j{}^t H_t{}^h = 0,$$
$$K_j{}^t f_t{}^h - f_j{}^t K_t{}^h = 0,$$
$$L_j{}^t f_t{}^h - f_j{}^t L_t{}^h = 0,$$

or equivalently,

$$H_{jt} f_i{}^t + H_{it} f_j{}^t = 0, \quad K_{jt} f_i{}^t + K_{it} f_j{}^t = 0, \quad L_{jt} f_i{}^t + L_{it} f_j{}^t = 0. \tag{2.1}$$

Transvecting the first equation of (2.1) with $f_k{}^i$, we have

$$H_{jt}(-\delta_i^t + p_i p^t) + H_{kt} f_j{}^t f_i{}^k = 0$$

from which, taking the skew-symmetric part of this

$$(H_{jt} p^t) p_i - (H_{it} p^t) p_j = 0,$$

which shows that

$$H_{jt} p^t = \alpha p_j, \tag{2.2}$$

where we have put $\alpha = H_{ji} p^j p^i$.

Differentiating (2.2) covariantly along M^{2n+1}, we have

$$(\nabla_k H_{jt}) p^t + H_{jt} \nabla_k p^t = (\nabla_k \alpha) p_j + \alpha \nabla_k p_j, \tag{2.3}$$

from which, taking the skew-symmetric part with respect to the indices k and j,

$$(\nabla_k H_{jt} - \nabla_j H_{kt}) p^t + H_j{}^t \nabla_k p_t - H_k{}^t \nabla_j p_t = (\nabla_k \alpha) p_j - (\nabla_j \alpha) p_k + \alpha (\nabla_k p_j - \nabla_j p_k). \tag{2.4}$$

Taking account of (1.27), (1.37), (1.30), (1.31) and (2.1), we have

$$\frac{c}{2} f_{jk} - H_j{}^t H_{ks} f_t{}^s + H_k{}^t H_{js} f_t{}^s + 2 l_j m_k - 2 m_j l_k = (\nabla_k \alpha) p_j - (\nabla_j \alpha) p_k - 2\alpha H_{kt} f_j{}^t. \tag{2.5}$$

If we take the symmetric part of (1.28) and use the last relationship of (2.1), we obtain

$$2K_{ji} = -(m_j p_i + m_i p_j). \tag{2.6}$$

In the same way we have from (1.29)

$$2L_{ji} = l_j p_i + l_i p_j. \tag{2.7}$$

Transvecting (2.6) and (2.7) with p^i and taking account of (1.32) and (1.33), we get

$$m_j = (m_t p^t) p_j = -K p_j, \tag{2.8}$$

$$l_j = (l_t p^t) p_j = L p_j. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.6) and (2.7) respectively, we have

$$K_{ji} = K p_j p_i, \tag{2.10}$$

$$L_{ji} = L p_j p_i. \tag{2.11}$$

Substitution of (2.8) and (2.9) into (2.5) gives

(2. 12) $$\frac{c}{2}f_{jk}-2H_j{}^tH_{ks}f_t{}^s=(\nabla_k\alpha)p_j-(\nabla_j\alpha)p_k-2\alpha H_{kt}f_j{}^t.$$

If we transvect (2. 12) with p^j, we find

(2. 13) $$\nabla_k\alpha=Ap_k,$$

where we have put $A=p^t\nabla_t\alpha$.

By using (2. 13), (2. 12) is represented by

$$\frac{c}{4}f_{kj}+H_j{}^tH_{ks}f_t{}^s=\alpha H_{kt}f_j{}^t.$$

Transvecting $f_i{}^j$ to this equation and making use the first equation of (2. 1), we have

(2. 14) $$H_{jt}H_i{}^t=\alpha H_{ji}+\frac{c}{4}(g_{ji}-p_jp_i).$$

Differentiating (2. 13) covariantly, it follows that

$$\nabla_j\nabla_k\alpha=(\nabla_jA)p_k+A\nabla_jp_k,$$

from which

$$(\nabla_jA)p_k-(\nabla_kA)p_j+A(\nabla_jp_k-\nabla_kp_j)=0,$$

or, using (1. 27) and (2. 1)

(2. 15) $$(\nabla_jA)p_k-(\nabla_kA)p_j+2AH_{kt}f_j{}^t=0.$$

Transvecting (2. 15) with p^k yields

$$\nabla_jA=Bp_j,$$

B being $p^t\nabla_tA$.

Thus (2. 15) reduces to

$$AH_{kt}f_i{}^t=0.$$

Transvecting this equation with $f_j{}^i$, we have

(2. 16) $$A(H_{ji}-\alpha p_jp_i)=0.$$

Now we assume that $M^{2n+4}(c)$ is a complex projective space CP^{n+2}, that is, $c=4$. If A is not zero, $H_{ji}=\alpha p_jp_i$ holds. But this is impossible because of (2. 14). Thus, it follows that $A=0$. Consequently, we can see from (2. 13) that α is a constant.

We now calculate the covariant differentation of the second fundamental tensor H_{ji}.

Differentiating (2. 14) covariantly, we have

(2. 17) $$\begin{aligned}(\nabla_kH_{jt})H_i{}^t+H_j{}^t\nabla_kH_{it}-\alpha\nabla_kH_{ji}&=-(\nabla_kp_j)p_i-p_j\nabla_kp_i\\&=(H_{kt}f_j{}^t)p_i+(H_{kt}f_i{}^t)p_j.\end{aligned}$$

Taking account of the equation (1. 37) of Codazzi (2. 1), (2. 8), (2. 9), (2. 10) and (2. 11), we find

(2. 18) $$\nabla_kH_{ji}-\nabla_jH_{ki}=p_kf_{ji}-p_jf_{ki}-2p_if_{kj}.$$

Taking the skew-symmetric part of (2. 17) with respect to k, j, substituting above result and taking account of (2. 1) and (2. 2), we have

(2. 19) $$H_j{}^t\nabla_kH_{it}-H_k{}^t\nabla_jH_{it}=2(H_{kt}f_j{}^t)p_i+\alpha(p_kf_{ji}-p_jf_{ki}).$$

Exchanging the indices i and k in (2.19), we find

$$H_j{}^t\nabla_i H_{kt} - H_i{}^t\nabla_j H_{kt} = 2(H_{it}f_j{}^t)p_k + \alpha(p_i f_{jk} - p_j f_{ik}).$$

Substituting (2.18) into this equation, we find

$$(2.20)\qquad H_j{}^t\nabla_k H_{it} - H_i{}^t\nabla_k H_{jt} = (H_{it}f_k{}^t)p_j - (H_{jt}f_k{}^t)p_i + \alpha((p_j f_{ik} - p_i f_{jk}).$$

Adding (2.17) and (2.20), we find by making use of (1.27) and (2.1)

$$(2.21)\qquad 2H_j{}^t\nabla_k H_{it} - \alpha\nabla_k H_{ji} = 2(H_{kt}f_j{}^t)p_i + \alpha(p_j f_{ik} - p_i f_{jk}).$$

Differentiating (2.2) covariantly and taking account of (1.27), (2.1), (2.2) and (2.14), we obtain

$$(2.22)\qquad p^t\nabla_k H_{it} = f_{ik}.$$

Transvecting $H_t{}^j$ to (2.21) and using (2.14) and (2.22), we have

$$(2.23)\qquad \alpha H_j{}^t\nabla_k H_{it} + 2\nabla_k H_{ji} = (\alpha^2+2)f_{ik}p_j - 2f_{kj}p_i - \alpha(H_{jt}f_k{}^t)p_i.$$

Multiplying α and 2 to (2.21) and (2.23) respectively, we obtain by subtracting the resulting equations each other

$$(\alpha^2+4)\nabla_k H_{ji} = (\alpha^2+4)(f_{ik}p_j + f_{jk}p_i),$$

from which,

$$(2.24)\qquad \nabla_k H_{ji} = f_{ik}p_j + f_{jk}p_i.$$

Thus we have the following

THEOREM 2.1. *Let M^{2n+1} $(n>1)$ be a semi-invariant submanifold with the distinguished normal C^A of codimension 3 in a complex space form.*

If $H\circ f - f\circ H = 0$, $K\circ f - f\circ K = 0$, $L\circ f - f\circ L = 0$ are satisfied, then the covariant derivative of H_{ji} is of the form

$$\nabla_k H_{ji} = f_{ik}p_j + f_{jk}p_i.$$

3. Submanifolds of codimension 3 in an even-dimensional Euclidean space satisfying $\lambda^2+\mu^2+\nu^2=1$.

In this section, we assume that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on a submanifold M^{2n+1} of codimension 3 of an even-dimensional Euclidean space E^{2n+4} satisfying $\lambda^2+\mu^2+\nu^2=1$, and suppose that the condition (2.1) holds. Then (1.36) – (1.42) with $c=0$ are valid because the ambient manifold is Euclidean.

From the condition (2.1) we have (2.2) – (2.4) and (2.8) – (2.11). Substituting (2.8) – (2.11) into (1.37) with $c=0$, we obtain

$$(3.1)\qquad \nabla_k H_{ji} - \nabla_j H_{ki} = 0.$$

If we take account of (1.27) and (3.1), then (2.4) reduces to

$$-2H_j{}^t H_{ks} f_t{}^s + 2(l_j m_k - m_j l_k) = (\nabla_k\alpha)p_j - (\nabla_j\alpha)p_k - \alpha(H_{kt}f_j{}^t - H_{jt}f_k{}^t),$$

or, using (2.1), (2.8) and (2.9)

(3.2) $$2H_j{}^tH_{ts}f_k{}^s=(\nabla_k\alpha)p_j-(\nabla_j\alpha)p_k+2\alpha H_{jt}f_k{}^t.$$

Transvecting p^j to (3.2), we find

(3.3) $$\nabla_k\alpha=Ap_k,$$

where we have put $A=p^t\nabla_t\alpha$.

Thus, (3.2) becomes

$$H_j{}^tH_{ts}f_k{}^s=\alpha H_{jt}f_k{}^t.$$

Transvecting this with $f_i{}^k$ and making use of (2.2), we find

(3.4) $$H_j{}^tH_{it}=\alpha H_{ji}.$$

Using the same method as that used to derive (2.16) from (2.3), we can derive from (3.3) the following:

$$A(H_{ji}-\alpha p_jp_i)=0.$$

If $A\neq 0$, then we have $H_{ji}=\alpha p_jp_i$. So we see from (1.27) that p^h is a parallel vector field and consequently

(3.5) $$K_{kji}{}^hp^i=0.$$

Assuming that the submanifold M^{2n+1} is locally irreducible, such a fact can never occur and thus it can never be that $A\neq 0$. Therefore, (3.3) means that α is a constant on M^{2n+1}.

Using the quite same method as that used to derive (2.24) from (2.14), we can derive from (3.4) the following equation:

(3.6) $$\nabla_kH_{ji}=0$$

because of (3.1) and the fact that α=constant. Since M^{2n+1} is locally irreducible, this implies that

(3.7) $$H_{ji}=Bg_{ji}$$

for a certain scalar B. If $B=0$, then we see from (1.27) and (2.7) that p^h is parallel and hence (3.5) is satisfied. Thereby, it can never be that $B=0$.

From (3.4) and (3.7), we have $B^2=\alpha B$ and consequently $B=\alpha\neq 0$. Thus (3.7) reduces to

(3.8) $$H_{ji}=\alpha g_{ji}$$

for some nonzero constant α. Substitution of this into (1.27) yields

(3.9) $$\nabla_jp_i=\alpha f_{ji}.$$

On the other hand, if we take account of (2.2), (2.8), (2.9) and (2.10), then (1.40) can be written as

$$\nabla_k(Lp_j)-\nabla_j(Lp_k)-K(n_jp_k-n_kp_j)=0,$$

or, as

(3.10) $$(\nabla_kL)p_j-(\nabla_jL)p_k+2L\alpha f_{kj}-K(n_jp_k-n_kp_j)=0$$

because of (3.9).

Transvecting p^j to (3.10), we find

$$\nabla_k L = -Kn_k + (p^t \nabla_t L + Kp^t n_t) p_k.$$

Thus, (3.10) becomes $L\alpha f_{kj}=0$, from which we obtain $L=0$ by the fact that $\alpha \neq 0$.

In the same way we see from (1.41) that $K=0$. Consequently the submanifold becomes pseudo-umbilical because of (3.8) and the fact that $L=K=0$.

According to Theorem A in §0, we conclude the following

THEOREM 3.1. *Let M^{2n+1} be a locally irreducible submanifold of codimension 3 of a Euclidean space E^{2n+4} with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If the second fundamental tensors commute with f, then M^{2n+1} is an intersection of a complex cone with generator as the distinguished normal C^A and $(2n+3)$-dimensional sphere.*

4. Submersion $\tilde{\pi} : S^{2m+1} \longrightarrow CP^m$ and immersion $i : M^{2m-3} \longrightarrow CP^m$.

In this section, we put $m=n+2$ $(n>1)$. Let $S^{2m+1}(1)$ be a hypersphere of radius 1 in an $(m+1)$-dimensional complex space C^{m+1}. We identify C^{m+1} with a real $(2m+2)$-dimensional space R^{2m+2} naturally. For briefness, we denote by S^{2m+1} above defined hypersphere $S^{2m+1}(1)$. Let $\tilde{\pi} : S^{2m+1} \longrightarrow CP^m$ be the natural projection of S^{2m+1} onto CP^m by the Hopf-fibration. We consider a Riemannian submersion $\pi : \bar{M}^{2m-2} \longrightarrow M^{2m-3}$ compatible with the Hopf-fibration $\tilde{\pi} : S^{2m+1} \longrightarrow CP^m$, $\bar{M}^{2m-2}$ being $\tilde{\pi}^{-1}(M^{2m-3})$. Also, we denote by $\bar{M}$ and M respectively above $\bar{M}^{2m-2}$ and M^{2m-3}. More precisely speaking, $\pi : \bar{M} \longrightarrow M$ is a Riemannian submersion with totally geodesic fibres such as the following diagram,

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\ i^*\ } & S^{2m+1} \\ \pi \big\downarrow & & \big\downarrow \tilde{\pi} \\ M & \xrightarrow{\ i\ } & CP^m \end{array}$$

where $i^* : \bar{M} \longrightarrow S^{2m+1}$ and $i : M \longrightarrow CP^m$ are isometric immersions. Let S^{2m+1} be covered by a system of coordinate neighborhoods $\{U'; y^\kappa\}$ such that $\tilde{\pi}(U')=U''$ are coordinate neighborhoods of CP^m with the local coordinate system (x^A).

Then we represent the natural projection $\tilde{\pi} : S^{2m+1} \longrightarrow CP^m$ by

(4.1) $$x^A = x^A(y^\kappa)$$

and we put

(4.2) $$E_\kappa{}^A = \partial_\kappa x^A \quad (\partial_\kappa = \partial / y^\kappa),$$

where the matrix $(E_\kappa{}^A)$ has the maximal rank.

Let $\tilde{\xi}^\kappa$ be the unit Sasakian structure vector field in S^{2m+1}. Since $\tilde{\xi}^\kappa$ is always tangent to the fibre $\tilde{\pi}^{-1}(\tilde{p})$, $\tilde{p}\in CP^m$, $\{F_\kappa{}^A, \tilde{\xi}_\kappa\}$ construct a local coframe in S^{2m+1}, where $\tilde{\xi}_\kappa=g_{\kappa\mu}\tilde{\xi}^\mu$ and $g_{\kappa\mu}$ denote the metric tensor of S^{2m+1}. We denote by $\{E^\kappa{}_A, \tilde{\xi}^\kappa\}$ the frame corresponding to this coframe. We have

(4.3) $$E_\kappa{}^A E^\kappa{}_B=\delta_B{}^A,\quad F_\kappa{}^A\tilde{\xi}^\kappa=0 \quad\text{and}\quad \tilde{\xi}_\kappa E^\kappa{}_A=0.$$

We now take coordinate neighborhoods $\{\bar{U}\,;\, y^\alpha\}$ of $\bar{M}$ such that $\pi(\bar{U})=U$ are coordinate neighborhoods of M with local coordinate system (x^h). Let the isometric immersion i^* and i be locally expressed by $y^\kappa=y^\kappa(y^\alpha)$ and $x^A=x^A(x^h)$ respectively. By the commutativity of the diagram, $\tilde{\pi}\circ i^*=i\circ\pi$ implies

$$x^A(y^\kappa(y^\alpha))=x^A(x^h(y^\alpha)),$$

where π is expressed locally by $x^h=x^h(y^\alpha)$. Hence we find that

(4.4) $$B_h{}^A E_\alpha{}^h=E_\kappa{}^A B_\alpha{}^\kappa,$$

where we have put $B_\alpha{}^\kappa=\partial_\alpha y^\kappa$ and $E_\alpha{}^h=\partial_\alpha x^h$. Let the indices $\kappa, \mu, \nu, \lambda, \cdots$, $A, B, C, D, \cdots, \alpha, \beta, \gamma, \delta, \cdots$, and $i, j, h, k, \cdots$ run over the ranges $\{1, 2, \cdots, 2m+1\}$, $\{1, 2, \cdots, 2m\}$, $\{1, 2, \cdots, 2m-2\}$ and $\{1, 2, \cdots, 2m-3\}$ respectively.

For an arbitrary point $P\in M$, we can choose the mutually orthogonal unit normal vector fields C^A, D^A and E^A to M defined in a neighborhood U of P such that $\{B_i{}^A, C^A, D^A, E^A\}$ spans the tangent space of CP^m at $i(P)$. Let P be an arbitrary point of the fibre $\pi^{-1}(P)$ over P. Then the horizontal lift C^κ, D^κ, and E^κ of C^A, D^A and E^A in such a way that $C^\kappa=C^A E^\kappa{}_A$, $D^\kappa=D^A E^\kappa{}_A$ and $E^\kappa=E^A E^\kappa{}_A$ are mutually orthogonal unit normal to $\bar{M}$ defined in the tubular neighborhood over $\bar{U}$ because of (4.4).

Because of $\tilde{\xi}^\kappa E_\kappa{}^A=0$, we may put

(4.5) $$\tilde{\xi}^\kappa=\xi^\alpha B_\alpha{}^\kappa,$$

where ξ^α is a local vector field in $\bar{M}$. (4.4) and (4.5) imply

(4.6) $$\xi_\alpha\xi^\alpha=1,\quad \xi^\alpha E_\alpha{}^h=0,$$

where $\xi_\alpha=g_{\beta\alpha}\xi^\beta$ and $g_{\beta\alpha}$ is the Riemannian metric tensor of $\bar{M}$ induced by $g_{\kappa\mu}$. Therefore, $\{E_\alpha{}^h, \xi_\alpha\}$ constructs a local coframe in M corresponding to $\{E_\kappa{}^A, \tilde{\xi}_\kappa\}$ in S^{2m+1}.

Denoting by $\{E^\alpha{}_h, \xi^\alpha\}$ the frame corresponding to this coframe, we have

(4.7) $$E_\alpha{}^h E^\alpha{}_k=\delta_k{}^h,\quad \xi_\alpha E^\alpha{}_h=0,\quad E_\alpha{}^j E^\beta{}_j+\xi_\alpha\xi^\beta=\delta_\alpha{}^\beta.$$

Then (4.4) and (4.6) imply

(4.8) $$E^\kappa{}_A B_h{}^A=B_\alpha{}^\kappa E^\alpha{}_h.$$

We denote by $\{{}_\mu{}^\lambda{}_\nu\}$, $\{{}_C{}^A{}_B\}$, $\{{}_\beta{}^\alpha{}_\gamma\}$ and $\{{}_j{}^k{}_i\}$ the Christoffel symbols formed with the Riemannian metrics $g_{\mu\lambda}$, g_{CB}, $g_{\beta\alpha}$ and g_{ji} respectively. Now we put

$$D_\mu E_\lambda{}^A=\partial_\mu E_\lambda{}^A-\{{}_\mu{}^\kappa{}_\lambda\}E_\kappa{}^A+\{{}_C{}^A{}_B\}E_\mu{}^C E_\lambda{}^B,$$
$$D_\mu E^\lambda{}_A=\partial_\mu E^\lambda{}_A+\{{}_\mu{}^\lambda{}_\kappa\}E^\kappa{}_A-\{{}_C{}^B{}_A\}E_\mu{}^C E^\lambda{}_B.$$

and

$$\bar{\nabla}_\beta E_\alpha{}^h = \partial_\beta E_\alpha{}^h - \{{}_\beta{}^\gamma{}_\alpha\} E_\gamma{}^h + \{{}_j{}^h{}_k\} E_\beta{}^j E_\alpha{}^k,$$
$$\bar{\nabla}_\beta E^\alpha{}_h = \partial_\beta E^\alpha{}_h + \{{}_\beta{}^\alpha{}_\gamma\} E^\gamma{}_h - \{{}_j{}^k{}_h\} E_\beta{}^j E^\alpha{}_k.$$

Since the metrics $g_{\lambda\mu}$ and $g_{\alpha\beta}$ are both invariant with respect to the submersions $\tilde{\pi}$ and π respectively, the van der Waerden Bortolotti covariant derivatives of $E_\lambda{}^A$, $E^\lambda{}_A$, and $E_\alpha{}^h$ $E^\alpha{}_h$ are given by

(4.9) $$D_\mu E_\lambda{}^A = h_B{}^A(E_\mu{}^B \tilde{\xi}_\lambda + E_\lambda{}^B \tilde{\xi}_\mu),$$
$$D_\mu E^\lambda{}_A = h_{BA} E_\mu{}^B \tilde{\xi}^\lambda - h_A{}^B \tilde{\xi}_\mu E^\lambda{}_B.$$

(4.10) $$\bar{\nabla}_\beta E_\alpha{}^h = h_j{}^h(E_\beta{}^j \xi_\alpha + \xi_\beta E_\alpha{}^j),$$
$$\bar{\nabla}_\beta E^\alpha{}_h = h_{jh} E_\beta{}^j \xi^\alpha - h_h{}^j \xi_\beta E^\alpha{}_j,$$

respectively, where $h_B{}^A = g^{AC} h_{BC}$, $h_j{}^h = g^{hi} h_{ji}$, h_{ji} are the structure tensors induced from the submersions $\tilde{\pi}$ and π respectively ([4]).

On the other hand, the equations of Gauss and Weingarten for the immersion $i^* : \bar{M} \longrightarrow S^{2m+1}$ are given by

(4.11) $$\bar{\nabla}_\beta B_\alpha{}^\kappa = \partial_\beta B_\alpha{}^\kappa + \{{}_\mu{}^\kappa{}_\nu\} B_\beta{}^\mu B_\alpha{}^\nu - \{{}_\beta{}^\gamma{}_\alpha\} B_\gamma{}^\kappa = H_{\beta\alpha} C^\kappa + K_{\beta\alpha} D^\kappa + L_{\beta\alpha} E^\kappa.$$
$$\bar{\nabla}_\beta C^\kappa = -H_\beta{}^\alpha B_\alpha{}^\kappa + l_\beta D^\kappa + m_\beta E^\kappa,$$
$$\bar{\nabla}_\beta D^\kappa = -K_\beta{}^\alpha B_\alpha{}^\kappa - l_\beta C^\kappa + n_\beta E^\kappa,$$
$$\bar{\nabla}_\beta E^\kappa = -L_\beta{}^\alpha B_\alpha{}^\kappa - m_\beta C^\kappa - n_\beta D^\kappa,$$

where $H_\beta{}^\alpha = H_{\beta\gamma} g^{\gamma\alpha}$, $K_\beta{}^\alpha = K_{\beta\gamma} g^{\gamma\alpha}$, $L_\beta{}^\alpha = L_{\beta\gamma} g^{\gamma\alpha}$, $H_{\beta\gamma}, K_{\beta\gamma}$ and $L_{\beta\gamma}$ are the second fundamental tensors with respect to C^κ, D^κ and E^κ respectively and $l_\beta, m_\beta, n_\beta$ are the third fundamental tensors. Moreover in a case of (4.4) and (4.8), we find

$$\nabla_j = E^\alpha{}_j \bar{\nabla}_\alpha.$$

We now put $F_\mu{}^\lambda = D_\mu \tilde{\xi}^\lambda$. The definition of a Sasakian structure induces

(4.12) $$F_\mu{}^\lambda F_\kappa{}^\mu = -\delta_\kappa{}^\lambda + \tilde{\xi}_\kappa \tilde{\xi}^\lambda, \quad F_\mu{}^\lambda \tilde{\xi}^\mu = 0, \quad \tilde{\xi}_\lambda F_\mu{}^\lambda = 0, \quad F_{\mu\lambda} + F_{\lambda\mu} = 0$$

and

(4.13) $$D_\mu F_\lambda{}^\kappa = \tilde{\xi}_\lambda \delta_\mu{}^\kappa - \tilde{\xi}^\kappa g_{\mu\lambda}, \quad D_\mu \tilde{\xi}^\kappa = F_\mu{}^\kappa,$$

where $F_{\mu\lambda} = F_\mu{}^\gamma g_{\lambda\gamma}$. Denoting by $\mathcal{L}$ the Lie differentiation with respect to $\tilde{\xi}$, we find

(4.14) $$\mathcal{L} F_\mu{}^\lambda = 0.$$

Putting

(4.15) $$F_B{}^A = F_\mu{}^\lambda E^\mu{}_B E_\lambda{}^A,$$

then we can see that $F_B{}^A$ defines a global tensor field of the same type as that of $F_\mu{}^\lambda$, which will be denoted by the same letter, with the aid of (4.14) and $\mathcal{L} E^\lambda{}_A = \mathcal{L} E_\lambda{}^A = 0$ ([4]). Moreover calculating the van der Waerden Bortolotti covariant derivative of the second equation of (4.3) and making use of (4.3), (4.9), the second equation of (4.13) and (4.15), we have

(4.16) $$F_B{}^A = -h_B{}^A,$$

which satisfies

(4.17) $$F_B{}^A F_C{}^B = -\delta_C{}^A$$

because of (4.12).

Differentiating (4.15) covariantly along CP^m and making use of (4.3), (4.9), (4.13), (4.15) and (4.16), we have

(4.18) $$\tilde{\nabla}_C F_B{}^A=0,$$

where $\tilde{\nabla}$ denotes the projection of D.

Therefore, the base space CP^m admits a Kaehlerian structure $\{F_B{}^A, g_{CB}\}$ represented by the structure tensor $h_B{}^A$ of the submersion $\tilde{\pi}: S^{2m+1} \longrightarrow CP^m$ defined by the Hopf-fibration. Let $K_{\kappa\mu\nu}{}^\lambda$ and $K_{DCB}{}^A$ be Riemann-Christoffel curvature tensors of S^{2m+1} and CP^m respectively.

The equations of co-Gauss are of the form

$$K_{DCB}{}^A=K_{\kappa\mu\nu}{}^\lambda E^\kappa{}_D E^\mu{}_C E^\nu{}_B E_\lambda{}^A+h_D{}^A h_{CB}-h_C{}^A h_{DB}-2h_{DC}h_B{}^A.$$

Since S^{2m+1} is a space of constant curvature 1, this equations reduce to

$$K_{DCB}{}^A=\delta_D{}^A g_{CB}-\delta_C{}^A g_{DB}+F_D{}^A F_{CB}-F_C{}^A F_{DB}-2F_{DC}F_B{}^A$$

because of (4.16).

Hence CP^m is a Kaehlerian manifold with constant holomorphic sectional curvature 4. Putting

(4.19) $$F_B{}^A B_i{}^B=f_i{}^h B_h{}^A+p_i C^A,$$
$$F_B{}^A C^B=-p^h B_h{}^A,\quad F_B{}^A D^B=-E^A,\quad F_B{}^A E^B=D^A,$$

as already shown in section 1, we can find the algebraic relations (1.12) $-$(1.15) and structure equations (1.26)$-$(1.33) with the distinguished normal C^A and $c=4$.

We put in each neighborhood U of M

(4.20) $$f_\beta{}^\alpha=f_i{}^h E_\beta{}^i E^\alpha{}_h,\quad p_\alpha=p_i E_\alpha{}^i,\quad p^\alpha=p^j E^\alpha{}_j.$$

Then, if we use (4.4), (4.8), (4.15), (4.19), (4.20) and $C^\kappa=C^A E^\kappa{}_A$, $D^\kappa=D^A E^\kappa{}_A$, $E^\kappa=E^A E^\kappa{}_A$, we obtain by the straightforward computation

(4.21) $$F_\mu{}^\kappa B_\alpha{}^\mu=f_\alpha{}^\beta B_\beta{}^\kappa+p_\alpha C^\kappa,$$
$$F_\mu{}^\kappa C^\mu=-p^\alpha B_\alpha{}^\kappa,\quad F_\mu{}^\kappa D^\mu=-E^\kappa,\quad F_\mu{}^\kappa E^\mu=D^\kappa.$$

Transvecting $F_\kappa{}^\nu$ to (4.21) and making use of (4.12) and (4.21), we can easily obtain the following relations:

(4.22) $$f_\alpha{}^\gamma f_\gamma{}^\beta=-\delta_\alpha{}^\beta+\xi_\alpha\xi^\beta+p_\alpha p^\beta,\quad f_\alpha{}^\beta p_\beta=0,\quad f_\alpha{}^\beta p^\alpha=0,$$
$$p_\alpha p^\alpha=1,\quad p_\alpha\xi^\alpha=0,\quad f_\alpha{}^\beta\xi^\alpha=0,\quad f_\alpha{}^\beta\xi_\beta=0,$$

where we have put $f_{\alpha\beta}=f_\alpha{}^\gamma g_{\gamma\beta}$.

Applying the operator $\bar{\nabla}_\gamma=B_\gamma{}^\mu D_\mu$ to (4.21) and using (4.11), (4.13) and (4.21), we find

(4.23) $$\bar{\nabla}_\gamma f_\beta{}^\alpha=\xi_\beta\delta_\gamma{}^\alpha-g_{\gamma\beta}\xi^\alpha-H_{\gamma\beta}p^\alpha+p_\beta H_\gamma{}^\alpha,$$

(4.24) $$\bar{\nabla}_\beta p_\alpha=-H_{\beta\gamma}f_\alpha{}^\gamma,\quad \bar{\nabla}_\beta p^\alpha=f_\beta{}^\gamma H_\gamma{}^\beta,$$

(4.25) $$K_{\beta\alpha}=-L_{\beta\gamma}f_\alpha{}^\gamma-m_\beta p_\alpha,$$

(4.26) $$L_{\beta\alpha}=K_{\beta\gamma}f_\alpha{}^\gamma+l_\beta p_\alpha,$$

(4.27) $$K_{\beta\alpha}p^\alpha=-m_\beta.$$

(4.28) $$L_{\beta\alpha}p^\alpha=l_\beta.$$

(4.25) and (4.26) imply that

$$K_\beta{}^\beta = -(m_\beta p^\beta) \tag{4.29}$$

and

$$L_\beta{}^\beta = l_\beta p^\beta. \tag{4.30}$$

Also, if we apply the operator $\bar{\nabla}_\beta$ to (4.5) and take account of (4.11) and (4.13), we have

$$\bar{\nabla}_\gamma \xi^\alpha = f_\gamma{}^\alpha, \quad H_{\gamma\beta}\xi^\beta = p_\gamma, \quad K_{\gamma\beta}\xi^\beta = L_{\gamma\beta}\xi^\beta = 0. \tag{4.31}$$

Since the ambient manifold S^{2m+1} is a space of constant curvature 1, the equations of Gauss are given by

$$K_{\delta\gamma\beta}{}^\alpha = \delta_\delta^\alpha g_{\gamma\beta} - \delta_\gamma^\alpha g_{\delta\beta} + H_\delta{}^\alpha H_{\gamma\beta} - H_\gamma{}^\alpha H_{\delta\beta} + K_\delta{}^\alpha K_{\gamma\beta} - K_\gamma{}^\alpha K_{\delta\beta} + L_\delta{}^\alpha L_{\gamma\beta} - L_\gamma{}^\alpha L_{\delta\beta}, \tag{4.32}$$

$K_{\delta\gamma\beta}{}^\alpha$ being the Riemann-Christoffel curvature tensor of $\bar{M}$, those of Codazzi by

$$\bar{\nabla}_\delta H_{\gamma\beta} - \bar{\nabla}_\gamma H_{\delta\beta} - l_\delta K_{\gamma\beta} + l_\gamma K_{\delta\beta} - m_\delta L_{\gamma\beta} + m_\gamma K_{\delta\beta} = 0, \tag{4.33}$$

$$\bar{\nabla}_\delta K_{\gamma\beta} - \bar{\nabla}_\gamma K_{\delta\beta} + l_\delta H_{\gamma\beta} - l_\gamma H_{\delta\beta} - n_\delta L_{\gamma\beta} + n_\gamma L_{\delta\beta} = 0, \tag{4.34}$$

$$\bar{\nabla}_\delta L_{\gamma\beta} - \bar{\nabla}_\gamma L_{\delta\beta} + m_\delta H_{\gamma\beta} - m_\gamma H_{\delta\beta} + n_\delta K_{\gamma\beta} - n_\gamma K_{\delta\beta} = 0, \tag{4.35}$$

and those of Ricci by

$$\bar{\nabla}_\beta l_\alpha - \bar{\nabla}_\alpha l_\beta + H_\beta{}^\gamma K_{\alpha\gamma} - H_\alpha{}^\gamma K_{\beta\gamma} + m_\beta n_\alpha - m_\alpha n_\beta = 0, \tag{4.36}$$

$$\bar{\nabla}_\beta m_\alpha - \bar{\nabla}_\alpha m_\beta + H_\beta{}^\gamma L_{\alpha\gamma} - H_\alpha{}^\gamma L_{\beta\gamma} + n_\beta l_\alpha - n_\alpha l_\beta = 0, \tag{4.37}$$

$$\bar{\nabla}_\beta n_\alpha - \bar{\nabla}_\alpha n_\beta + K_\beta{}^\gamma L_{\alpha\gamma} - K_\alpha{}^\gamma L_{\beta\gamma} + l_\beta m_\alpha - l_\alpha m_\beta = 0. \tag{4.38}$$

On the other hand, the second relation of (4.6), (4.10), (4.22) and the first equation of (4.31) give

$$f_j{}^h = -h_j{}^h. \tag{4.39}$$

Then, applying the operator $\nabla_j = B_j{}^A \tilde{\nabla}_A = E^\alpha{}_j \bar{\nabla}_\alpha = B_j{}^B E^\varepsilon{}_B D_\varepsilon$ to (4.4) and using (1.22), (4.9), (4.10), (4.11), (4.16), (4.19) and (4.39), we have

$$(H_{jh}C^A + K_{jh}D^A + L_{jh}E^A)E_\alpha{}^h = (-p_j\xi_\alpha + H_{\gamma\alpha}E^\gamma{}_j)C^A + (K_{\gamma\alpha}D^A + L_{\gamma\alpha}E^A)E^\gamma{}_j,$$

from which, comparing the normal parts of both sides,

$$H_{jh}E_\alpha{}^h = H_{\gamma\alpha}E^\gamma{}_j - p_j\xi_\alpha, \quad K_{jh}E_\alpha{}^h = K_{\gamma\alpha}E^\gamma{}_j, \quad L_{jh}E_\alpha{}^h = L_{\gamma\alpha}E^\gamma{}_j. \tag{4.40}$$

If we transvect this with $E_\beta{}^j$ respectively and make use of (4.7), (4.20) and (4.31), then we get

$$H_{\beta\alpha} = H_{ji}E_\beta{}^j E_\alpha{}^i + p_\beta\xi_\alpha + p_\alpha\xi_\beta, \quad K_{\beta\alpha} = K_{ji}E_\beta{}^j E_\alpha{}^i, \quad L_{\beta\alpha} = L_{ji}E_\beta{}^j E_\alpha{}^i. \tag{4.41}$$

We now assume that the second fundamental tensors of the base space M commute with the structure tensor $f_j{}^h$, that is, (2.1) holds. Then we can easily verify that second fundamental tensors of the total space M commute with f because of (4.20) and (4.22), that is,

$$H_\beta{}^\gamma f_\gamma{}^\alpha - f_\beta{}^\gamma H_\gamma{}^\alpha = 0, \quad K_\beta{}^\gamma f_\gamma{}^\alpha - f_\beta{}^\gamma K_\gamma{}^\alpha = 0, \quad L_\beta{}^\gamma f_\gamma{}^\alpha - f_\beta{}^\gamma L_\gamma{}^\alpha = 0,$$

or equivalently,

(4. 42) $H_{\beta\gamma}f_\alpha{}^\gamma+H_{\alpha\gamma}f_\beta{}^\gamma=0,\quad K_{\beta\gamma}f_\alpha{}^\gamma+K_{\alpha\gamma}f_\beta{}^\gamma=0,\quad L_{\beta\gamma}f_\beta{}^\gamma+L_{\alpha\gamma}f_\beta{}^\gamma=0.$

Transvecting $H_\gamma{}^\alpha$ to the first equation of (4. 41) and using (2. 2) and (4. 22), we find

$$H_{\beta\alpha}H_\gamma{}^\alpha=H_{jt}H_i{}^tE_\beta{}^jE_\gamma{}^i+\alpha(p_\beta\xi_\gamma+p_\gamma\xi_\beta)+p_\beta p_\gamma+\xi_\beta\xi_\gamma,$$

from which, substituting (2. 14),

$$H_{\beta\alpha}H_\gamma{}^\alpha=\alpha H_{\beta\gamma}+g_{\beta\gamma}$$

because of (4. 7), (4. 20) and (4. 41). Thus we have the following

LEMMA 4. 1. *Let M be a submanifold of codimension 3 of a complex projective space with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If the second fundamental tensor H_{ji} commutes with f_{ji}, then the eigenpolynomial of the second fundamental tensor $H_{\beta\alpha}$ of the total space $\overline{M}$ over M is given by*

(4. 43) $$H_{\gamma\alpha}H_\beta{}^\alpha=\alpha H_{\gamma\beta}+g_{\gamma\beta}.$$

Also, transvecting the first equation of (4. 41) with p^α and utilizing (2. 2) (4. 20) and (4. 22), we find

(4. 44) $$H_{\beta\alpha}p^\alpha=\alpha p_\beta+\xi_\beta.$$

On the other hand, substituting (2. 10) and (2. 11) into the second and third equation of (4. 41) respectively, we obtain

(4. 45) $$K_{\beta\alpha}=Kp_\beta p_\alpha,$$

(4. 46) $$L_{\beta\alpha}=Lp_\beta p_\alpha.$$

Substituting (4. 45) and (4. 46) into (4. 27) and (4. 28) respectively, we find

(4. 47) $$m_\beta=-Kp_\beta,$$

(4. 48) $$l_\beta=Lp_\beta$$

because of (4. 22).

We next prove the following

LEMMA 4. 2. *Let M be a submanifold of codimension 3 of a complex projective space CP^m $(m>2)$ with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If $H_j{}^tf_t{}^h-f_j{}^tH_t{}^h=0$, $K_j{}^tf_t{}^h-f_j{}^tK_t{}^h=0$ and $L_j{}^tf_t{}^h-f_j{}^tL_t{}^h=0$ hold, then we have*

$$K_{ji}=L_{ji}=0,\quad K_{\beta\alpha}=L_{\beta\alpha}=0.$$

Proof. Differentiating (4. 45) covariantly and using (4. 24), we find

$$\bar\nabla_\gamma K_{\beta\alpha}=(\bar\nabla_\gamma K)p_\beta p_\alpha-K(H_{\gamma\varepsilon}f_\beta{}^\varepsilon p_\alpha+H_{\gamma\varepsilon}f_\alpha{}^\varepsilon p_\beta),$$

from which, taking the skew-symmetric part with respect to the indices γ and β,

$$l_\beta H_{\gamma\alpha}-l_\gamma H_{\beta\alpha}+n_\gamma L_{\beta\alpha}-n_\beta L_{\gamma\alpha}$$
$$=\{(\bar\nabla_\gamma K)p_\beta-(\bar\nabla_\beta K)p_\gamma\}p_\alpha-K(2H_{\gamma\varepsilon}f_\beta{}^\varepsilon p_\alpha+H_{\gamma\varepsilon}f_\alpha{}^\varepsilon p_\beta-H_{\beta\varepsilon}f_\alpha{}^\varepsilon p_\gamma)$$

because of (4. 34) and (4. 42), or, using (4. 46) and (4. 48),

(4. 49) $$L(p_\beta H_{\gamma\alpha}-p_\gamma H_{\beta\alpha}+n_\gamma p_\beta p_\alpha-n_\beta p_\gamma p_\alpha)$$

$$=\{(\bar{\nabla}_\gamma K)p_\beta-(\bar{\nabla}_\beta K)p_\gamma\}p_\alpha-K(2H_{\gamma\varepsilon}f_\beta{}^\varepsilon p_\alpha+H_{\gamma\varepsilon}f_\alpha{}^\varepsilon p_\beta-H_{\beta\varepsilon}f_\alpha{}^\varepsilon p_\gamma).$$

Transvecting this equation with $p^\beta p^\alpha$ yields

$$\bar{\nabla}_\gamma K=L(\xi_\gamma+n_\gamma)+\{p^\beta\bar{\nabla}_\beta K-L(n_\beta p^\beta)\}p_\gamma$$

because of (4.22) and (4.44). Consequently (4.49) becomes

$$L(p_\beta H_{\gamma\alpha}-p_\gamma H_{\beta\alpha})$$
$$=L(\xi_\gamma p_\beta-\xi_\beta p_\gamma)p_\alpha-K(2H_{\gamma\varepsilon}f_\beta{}^\varepsilon p_\alpha+H_{\gamma\varepsilon}f_\alpha{}^\varepsilon p_\beta-H_{\beta\varepsilon}f_\alpha{}^\varepsilon p_\gamma).$$

Transvecting this equation with p^α and taking account of (4.22) and (4.44), we find

$$KH_{\gamma\varepsilon}f_\beta{}^\varepsilon=0, \tag{4.50}$$

from which, transvecting with $f_\alpha{}^\beta$,

$$K(H_{\gamma\alpha}-p_\gamma\xi_\alpha-\xi_\gamma p_\alpha-\alpha p_\gamma p_\alpha)=0$$

because of (4.22), (4.31) and (4.44). If we take the trace of the matrix formed with the left member of the above equation, we see that

$$KH_\varepsilon{}^\varepsilon=K\alpha \tag{4.51}$$

because of (4.22).

Taking the square of the norm of the left member of (4.50) and using (4.6), (4.22), (4.31), (4.43) and (4.44), we find

$$K^2\{\alpha H_\varepsilon{}^\varepsilon+(2m-2)-\alpha^2-2\}=0,$$

or using (4.51),

$$2(m-2)K^2=0.$$

Therefore, $K=0$ because of $m>2$ and hence we have $K_{\beta\alpha}=0$. Moreover, we have also $K_{ji}=0$ by virtue of (2.10).

By the same way, we obtain $L=0$, $L_{ji}=0$, $L_{\beta\alpha}=0$.
Thus we completes the proof of Lemma 4.2.

From Lemma 4.2 it follows that the submanifold M is contained in a manifold $M_0(\subset CP^m)$ as a hypersurface. Also, we can easily verify that M_0 is a $(2m-2)$-dimensional complex projective space as an invariant submanifold of CP^m because of (1.20) and (1.21), the total space $\bar{M}$ over M is contained in S^{2m-1} $(\subset S^{2m+1})$ satisfying $\tilde{\pi}\,|S^{2m-1}: S^{2m-1}\longrightarrow CP^m$.

The last lemma is as follows.

LEMMA 4.3. *Under the same assumptions as those stated in Lemma 4.2, we obtain*

$$\bar{\nabla}_\gamma H_{\beta\alpha}=0. \tag{4.52}$$

Proof. Applying the operator $\nabla_k=E^\gamma{}_k\bar{\nabla}_\gamma$ to the first equation of (4.41), we have

$$E^\gamma{}_k\bar{\nabla}_\gamma H_{\beta\alpha}=(\nabla_k H_{ji})E_\beta{}^jE_\alpha{}^i+H_{ji}E^\gamma{}_k(\bar{\nabla}_\gamma E_\beta{}^j)E_\alpha{}^i+H_{ji}E_\beta{}^jE^\gamma{}_k\bar{\nabla}_\gamma E_\alpha{}^i$$
$$+E^\gamma{}_k(\bar{\nabla}_\gamma p_\alpha)\xi_\beta+p_\alpha E^\gamma{}_k\bar{\nabla}_\gamma\xi_\beta+E^\gamma{}_k(\bar{\nabla}_\gamma p_\beta)\xi_\alpha+p_\beta E^\gamma{}_k\bar{\nabla}_\gamma\xi_\alpha.$$

Substituting (4.9) with $h_j{}^h=-f_j{}^h$, (4.24) and (4.31) into this equation, we find

$$E^{\gamma}{}_{k}\bar{\nabla}_{\gamma}H_{\beta\alpha}=(\nabla_{k}H_{ji}+p_{i}f_{ki}+p_{j}f_{ki})E_{\beta}{}^{j}E_{\alpha}{}^{i} - (H_{kj}f_{i}{}^{j}+H_{ij}f_{k}{}^{j})(E_{\beta}{}^{i}\xi_{\alpha}+E_{\alpha}{}^{i}\xi_{\beta})$$

because of (4.20) and (4.40), from which, using (2.1) and (2.24),

$$E^{\gamma}{}_{k}\bar{\nabla}_{\gamma}H_{\beta\alpha}=0. \tag{4.53}$$

On the other hand, by (4.45)−(4.48), (4.33) reduces to

$$\bar{\nabla}_{\gamma}H_{\beta\alpha}-\bar{\nabla}_{\beta}H_{\gamma\alpha}=0. \tag{4.54}$$

Transvecting (4.53) with $E_{\delta}{}^{k}$ and using (4.7), we get

$$\bar{\nabla}_{\delta}H_{\beta\alpha}=(\xi^{\gamma}\bar{\nabla}_{\gamma}H_{\beta\alpha})\xi_{\delta}.$$

Differentiating the second equation of (4.31) and taking account of (4.31), (4.24), (4.42) and (4.54), we have $\bar{\nabla}_{\delta}H_{\beta\alpha}=0$.

Thus the Lemma 4.3 is proved.

Denoting by x the eigenvalue corresponding to an eigenvector of $H_{\beta}{}^{\alpha}$, (4.43) implies $x^2-\alpha x-1=0$. Then we see that $H_{\beta}{}^{\alpha}$ has exactly two constant eigenvalues $x_1=\alpha+\sqrt{\alpha^2+4}/2$ and $x_2=\alpha-\sqrt{\alpha^2+4}/2$. Then we can obtain from (4.44) that

$$H_{\beta}{}^{\alpha}(x_1p^{\beta}+\xi^{\beta})=x_1(x_1p^{\alpha}+\xi^{\alpha}),$$

that is, $x_1p^{\alpha}+\xi^{\alpha}$ is an eigenvector of $H_{\beta}{}^{\alpha}$, which will be denoted by $e_1{}^{\alpha}$. Assuming that there exists another eigenvector $e_2{}^{\alpha}$ of $H_{\beta}{}^{\alpha}$ corresponding to x_1 and supposing that $e_2{}^{\alpha}$ is orthogonal to $e_1{}^{\alpha}$, we find

$$x_1(p_{\alpha}e_2{}^{\alpha})+\xi_{\alpha}e_2{}^{\alpha}=0. \tag{4.55}$$

From (4.31), we have

$$x_1(e_2{}^{\alpha}\xi_{\alpha})-p_{\alpha}e_2{}^{\alpha}=0 \tag{4.56}$$

because of $H_{\beta}{}^{\alpha}e_2{}^{\beta}=x_1e_2{}^{\alpha}$. The last two equations yield

$$\xi_{\alpha}e_2{}^{\alpha}=0,\quad p_{\alpha}e_2{}^{\alpha}=0. \tag{4.57}$$

From the first relationship of (4.42) we find

$$H_{\gamma}{}^{\beta}(f_{\alpha}{}^{\gamma}e_2{}^{\alpha})=x_1(f_{\alpha}{}^{\beta}e_2{}^{\alpha}).$$

Thus, $f_{\alpha}{}^{\beta}e_2{}^{\alpha}$ is also eigenvector of $H_{\beta}{}^{\alpha}$ corresponding to x_1, which is mutually orthogonal to $e_2{}^{\alpha}$ and to $e_1{}^{\alpha}$ because of (4.22) and (4.53). Therefore, the multiplicity of the eigenvalues of x_1 is necessary $2p+1$ for some integer p. In the same way, we can prove that the multiplicity of x_2 is $2q+1$, q being an integer such that $q=m-1-p$. From this fact and (4.52), the eigenspaces corresponding to x_1 and x_2 define respectively $(2p+1)$ and $(2q+1)$-dimensional distributions Dx_1 and Dx_2 over $\bar{M}$ and these two distributions are both integrable and parallel. Moreover each integral manifold of Dx_1 and Dx_2 is totally geodesic in $\bar{M}$ and totally umbilical in CP^m. Making use of a usual manner (cf. [6]), we obtain

$$\bar{M}=S^{2p+1}(a)\times S^{2q+1}(b),$$

(p, q) being some portion of $m-1$ and $a^2+b^2=1$.

Thus we have the following

THEOREM 4.4. *Let M be a complete submanifold of codimension 3 of a complex projective space CP^m $(m>2)$ with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If the second fundamental tensors of M commute with f, then M is of the form*

$$M=M_{p,q}{}^C(a, b),$$

(p, q) being some portion of $m-1$ and $a^2+b^2=1$.

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