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SOME ASPECT OF RINGS OF QUOTIENTS

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1. Introduction

Let R be an associative ring with an identity element 1, and ${}_R\mathfrak{M}$ denote the category of all left R-modules and R-homomorphisms. For a given torsion radical σ on ${}_R\mathfrak{M}$, the ring of quotients $Q_\sigma(R)$ of R w. r. t. σ is defined by a direct limit

$$Q_\sigma(R) = \varinjlim \mathrm{Hom}_R(U, R/\sigma(R)), \quad U \in \mathcal{J}_\sigma,$$

where $\mathcal{J}_\sigma$ is the associated (Gabriel) filter of σ ([1], [3], [4]).

The purpose of this article is to give a generalization of such a construction of rings of quotients and to study the structure of rings of quotients by means of the σ-injectivity and some other properties.

2. Some preliminaries

Let σ be a torsion radical [3] on ${}_R\mathfrak{M}$, and $\mathcal{J}_\sigma$ be its associated filter. A left R-module M is called *σ-torsion* (*σ-torsion free*) if $\sigma(M)=M$ (resp. $\sigma(M)=0$). A submodule N of M is called a *σ-submodule*, written by $N \triangleleft M$, if M/N is σ-torsion. A left R-module E is called *σ-injective* if, whenever $N \triangleleft M$ for a left R-module M, every homomorphism $f: N \to E$ can be extended to a homomorphism $g: M \to E$. Moreover, if the existence of such a g is unique, E is called to be *faithfully σ-injective.* For every σ-torsion free module M, there exists a unique (upto isomorphism) faithfully σ-injective module $E_\sigma(M)$ containing M as a σ-submodule. Such a module $E_\sigma(M)$ is called the *σ-injective hull* of M [4]. Throughout this note R will denote a ring with identity 1 and all modules are assumed to be left R-modules. We also assume that σ is a torsion radical and $\mathcal{J}_\sigma$ is its associated filter such that ${}_RR$ is σ-torsion free.

3. The ring of quotients

Before entering our main discussions, we prepare some lemmas such as

LEMMA 1. *Let N, M and K be R-modules such that $N < M$ and $M < K$.*

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Then $N \triangleleft M$ *if* $N \triangleleft K$.

The proof of this lemma is trivial, and is omitted.

LEMMA 2. *Let* M *be a faithfully* σ*-injective module. Then, for every* $U \in J_\sigma$, $(0 : U)_M = 0$, *where* $(0 : U)_M = \{x \in M \mid Ux = 0\}$.

Proof. Let $U \in \mathcal{J}_\sigma$, and $x \in (0 : U)_M$. Then, for the the homomorphism $u \mapsto ux : U \rightarrow M$, $r \mapsto rx : R \rightarrow M$ is evidenly an extension of it. But, since M is faithfully σ-injective, this homomorphism must be the zero homomorphism of R into M.

Since the ring of quotients $Q_\sigma(R)$ of R w.r.t. σ is faithfully σ-injective such that $R \triangleleft Q_\sigma(R)$ [1], $Q_\sigma(R)$ may be identified with $E_\sigma(R)$, the σ-in jective hull of R, not only as left R-modules but also as rings.

THEOREM 3. *Let* $U \in \mathcal{J}_\sigma$ *and* M *be a submodule of* $E_\sigma(R)$. *Then* $(M : U)_{E\sigma(R)} = \{q \in E_\sigma(R) \mid Uq \subset M\}$ *is isomorphic to* $\mathrm{Hom}_R(U, M)$ *canonically.*

Proof. Define $\phi : (M : U)_{E\sigma(R)} \rightarrow \mathrm{Hom}_R(U, M)$ by $\phi(q)(u) = uq$ for all $u \in U$. Suppose that $\phi(q) = 0$. Then $Uq = 0$ so that $q = 0$ by Lemma 2. Therefore ϕ is a monomorphism.

Now let $f : U \rightarrow M$ be any homomorphism. Since $U \triangleleft R$, there is a (unique) $g \in \mathrm{Hom}_R(R, E_\sigma(R))$ which coincides with f on U. Let $g(1) = q \in E_\sigma(R)$. Then $\phi(q) = f$ showing that ϕ is an epimorphism.

COROLLARY. *Let* $U \in \mathcal{J}_\sigma$ *and* $f \in \mathrm{Hom}_R(U, R)$. *Then there is a unique* $q \in E_\sigma(R)$ *such that* $f(u) = uq$ *for all* $u \in U$.

Here, we give a more general form of rings of quotients of R w.r.t. σ than that of Goldman, O [1].

DEFINITION. An overring Q of R will be called a *ring of quotients of* R w.r.t. σ if ${}_RR$ is a large σ-submodule of ${}_RQ$.

Evidently R itself is a ring of quotients of R w.r.t. σ. Since $Q_\sigma(R) = E_\sigma(R)$ is an essential extension of R and $R \triangleleft Q_\sigma(R)$. $Q_\sigma(R)$ is also a ring of quotients of R w.r.t. σ.

THEOREM 4. *If* Q *is a ring of quotients of* R w.r.t. σ, *then there is a unique monomorphism* ${}_RQ \rightarrow {}_RE_\sigma(R)$ *whose restriction to* R *is* id_R, *the identity homomorphism of* R. *Therefore any ring of quotients of* R w.r.t. σ *may be regarded as an* R*-submodule of* $E_\sigma(R)$.

Proof. For id_R, considered as a homomorphism $R \rightarrow E_\sigma(R)$, there exists a unique homomorphism $h : Q \rightarrow E_\sigma(R)$ such that $h(r) = r$ for all $r \in R$. For the submodule $\ker h$ of Q, we have that $\ker h \cap R = \ker id_R = 0$ so that ker $h=0$ since R is large in Q.

COROLLARY 1. *Let Q be a ring of quotients of R w.r.t. σ and let $U\in\mathcal{J}_\sigma$. Then, for each $q\in Q$, $(U:q)_R\in\mathcal{J}_\sigma$, where $(U:q)_R=\{r\in Q\mid rq\in U\}$.*

Proof. Clearly $(U:q)_R$ is a left ideal of R. Since $R\lhd E_\sigma(R)$ and $U<R$, we know that $U\lhd E_\sigma(R)$ [4]. For each $q\in Q$, $q\in E_\sigma(R)$ by Theorem 4. Let $r\in R$, then $rq\in E_\sigma(R)$. Thus there is a $B\in\mathcal{J}_\sigma$ such that $Brq\subset U$, i.e. $Br\subset(U:q)_R$

COROLLARY 2. *Any ring of quotients of R w.r.t. σ comes to be a subring of $E_\sigma(R)$.*

Proof. Use Lemma 2, Theorem 4 and its Corollary 1.

Let Q be any overring of R such that ${}_RQ$ is a submodule of $E_\sigma(R)$. Then $R\lhd Q$ by Lemma 1. Moreover, for any nonzero $q\in Q$, Rq is a nonzero submodule of $E_\sigma(R)$ so that $Rq\cap R\neq 0$ since $E_\sigma(R)$ is an essential extension of R. But this says that Q itself is an essential extension of R.
Thus we have

THEOREM 5. *An overring Q of R is a ring of quotients of R w.r.t. σ iff ${}_RQ$ is a submodule of $E_\sigma(R)$*

Then, when does a ring of quotients Q of R w.r.t. σ come to be isomorphic to $E_\sigma(R)$? An answer to this question is given here in

THEOREM 6. *An overring Q of R isomorphic to the ring $E_\sigma(R)$ under an isomorphism whose restriction to R is id_R iff*

(1) Q is a ring of quotients of R

and

(2) For every $U\in\mathcal{J}_\sigma$ and every $f\in\mathrm{Hom}_R(U,R)$, there exists a unique $q\in Q$ such that $f(u)=uq$ for all $u\in U$.

Proof. Suppose that $Q\approx E_\sigma(R)$. Then (1) is clear and (2) follows from the Corollary of Theorem 3.

For the converse, assume that Q is an overring of R for which (1) and (2) hold. Then, by (1) and Corollary 2 of Theorem 4, Q is a subring of $E_\sigma(R)$. Let $x\in E_\sigma(R)$ and let $U=(R:x)_R=\{r\in R\mid rx\in R\}$, then $U\in\mathcal{J}_\sigma$. Let $f\in\mathrm{Hom}_R(U,R)$ be the homomorphism $u\longmapsto ux$, then, by (2), there is a unique $q\in Q$ such that $ux=f(u)=uq$ for all $u\in U$. Therefore, $x=q$ so that $x\in Q$.

COROLLARY. *R is isomorphic to $E_\sigma(R)$ as rings iff for every $U\in\mathcal{J}_\sigma$ and every $f\in\mathrm{Hom}_R(U,R)$, there is a unique $r\in R$ such that $f(u)=ru$ for all $u\in U$.*

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