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ON THE GENERAL DEFINITION OF CONVOLUTION FOR SEVERAL DISTRIBUTIONS

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1. Introduction

This paper is a sequel to the paper of Youn and Richards [1]. In [1], we introduced a notion of convolvability for two distributions which, while it does not apply to arbitrary pairs of distributions, nevertheless has the virtue of including most of the classical situations where convolutions are used. Here we extend this definition to the case of n-fold convolutions.

Two of the properties which we would like to have for such a definition are associativity and commutativity. However, we find that if we proceed in the manner which is familiar from abstract algebra, then associativity does not hold. Namely, if the convolution $f_1*f_2*\cdots*f_n$ is defined as a concatenation of binary operations, say as $f_1*(f_2*(f_3*\cdots(f_{n-1}*f_n))\cdots)$, then the associative law fails. Examples showing this are given at the end of the paper. The way out of this difficulty is to define the convolution of n distributions as a single n-ary operation, rather than a succession of binary operations.

[A distinction similar to the one here, between binary and n-ary operations, occurs in Fubini's theorem in measure theory. There the "iterated integral" is distinguished from the "multiple integral", and Fubini's theorem is true only under the more restrictive condition that the multiple integral exists.]

The notations and terminology in this paper will follow those in [1]. Since these are fairly elaborate, we will not reproduce them here. However the following brief summary may be helpful.

$\mathscr{B}(R^k)$ denotes the space of C^∞ functions on R^k which are bounded together with all of their mixed partial derivatives. The topology on $\mathscr{B}(R^k)$ is the "strict topology" [1, Definition 2.1].

$\mathscr{B}'(R^k)$ is the dual of $\mathscr{B}(R^k)$. It may be represented as the space of all finite sums of derivatives of bounded complex measures on R^k [1, Proposition 2.3].

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There is a natural imbedding $\mathcal{D}(R^k)\subseteq\mathcal{B}(R^k)$ and $\mathcal{B}'(R^k)\subseteq\mathcal{D}'(R^k)$. Distributions $f\in\mathcal{B}'(R^k)$ are called *integrable.*

We observe that $\mathcal{B}(R^k)$ contains the constant function 1. If f is integrable, then we define $\int_{R^k} f(x)dx$ to be $\langle f, 1\rangle$.

If U and V are complementary subspaces of R^k, so that $R^k=U\oplus V$, and $f\in\mathcal{D}'(R^k)$ is an arbitrary distributionon R^k, then we say that f is *partially integrable over* V if, for each test function $\varphi\in\mathcal{D}(U)$, the product $\varphi(u)f(u, v)$ is an integrable distribution on R^k. We define $\int_V f(u, v)\,dv$ to be the distribution on U given by

$$\varphi(u)\to\langle\varphi(u)f(u, v), 1\rangle.$$

In [1], we establish a number of properties of the partial integral which will be used here. Among these are a "Fubini theorem" [1, Theorem 2.16] and a "Variable constants theorem" [1, Theorem 2.17]. These theorems are restated below.

2. Definitions and main results

DEFINITION. Let $f_1, \cdots, f_n\in\mathcal{D}'(R^k)$. We say $f_1*\cdots*f_n$ exists if, for any $\varphi\in\mathcal{D}(R^k)$, $\varphi(x_1+\cdots+x_n)f_1(x_1)\cdots f_n(x_n)\in\mathcal{B}'(R^{nk})$. In this case, we define

$$\langle f_1*\cdots*f_n, \varphi\rangle=\langle\varphi(x_1+\cdots+x_n)f_1(x_1)\cdots f_n(x_n),\ 1\rangle.$$

REMARK. Associativity will be a consequence of this definition, as long as all of the convolutions involved are defined.

PROPOSITION. *The n-fold convolution is commutative. That is, for any permutation $\sigma(1), \cdots, \sigma(n)$ of the integers $1, \cdots, n$:*

$$f_{\sigma(1)}*\cdots*f_{\sigma(n)}=f_1*\cdots*f_n.$$

REMARK. We have to consider arbitrary permutations (and not just the case $f*g=g*f$), because we do not yet have an associative law.

Proof of proposition. This follows at once from the commutativity of the tensor product.

THEOREM 1. *Let $f_1, \cdots, f_n\in\mathcal{D}'(R^k)$ and $g_1, \cdots, g_m\in\mathcal{E}'(R^k)$ (distributions with compact support). Then if $f_1*\cdots*f_n$ exists, so does $f_1*\cdots*f_n*g_1\cdots*g_m$.*

LEMMA 1. (Existence of a mesa function) *Let $g_1, \cdots, g_m\in\mathcal{E}'(R^k)$ and $\varphi\in\mathcal{D}(R^k)$, $m<n$. Then there is a "mesa function" $\theta\in\mathcal{D}(R^k)$* (i. e. *a function identically equal to* 1 *over a certain compact set*) *such that :*

$$g_1(x_1)\cdots g_m(x_m)\varphi(x_1+\cdots+x_m+\cdots+x_n)$$
$$=\theta(x_{m+1}+\cdots+x_n)g_1(x_1)\cdots g_m(x_m)\varphi(x_1+\cdots+x_m+\cdots x_n),$$

where each $x_i\in R^k$.

Proof. Let $a>0$ be such that the support of φ and that of each g_i lies in $\{|x| : |x|<a\}$. Now choose $\theta\in\mathcal{D}(R^k)$ so that $\theta(x)=1$ for $x\leq(m+1)a$. Then $\theta(x_{m+1}+\cdots+x_n)=1$ over the support of $g_1(x_1)\cdots g_m(x_m)\varphi(x_1+\cdots+x_m+\cdots+x_n)$. Hence the result follows. Q. E. D.

Proof of Theorem 1. By our basic definition, we have to prove that, for any $\varphi\in\mathcal{D}(R^k)$:

$$(*)\quad \varphi(x_1+\cdots+x_{n+m})f_1(x_1)\cdots f_n(x_n)g_1(x_{n+1})\cdots g_m(x_{n+m})\in\mathcal{B}'(R^{(n+m)k}).$$

We will use the facts a) any distribution with compact support belongs to $\mathcal{B}'$; b) for any distribution $f\in\mathcal{B}'$ and any function $\varphi\in\mathcal{B}$, the product $\varphi f\in\mathcal{B}'$; and c) tensor products of distributions of $\mathcal{B}'$ type are again of $\mathcal{B}'$ type [1, Propositions 2.5 and 2.6].

Now let $\varphi\in\mathcal{D}(R^k)$. Then by the above lemma, there is a $\theta\in\mathcal{D}(R^k)$ such that

$$g_1(x_{n+1})\cdots g_m(x_{n+m})\varphi(x_1+\cdots x_n+\cdots+x_{n+m})$$
$$=\theta(x_1+\cdots+x_n)g_1(x_{n+1})\cdots g_m(x_{n+m})\varphi(x_1+\cdots+x_{n+m}).$$

Hence

$$\varphi(x_1+\cdots+x_{n+m})f_1(x_1)\cdots f_n(x_n)g_1(x_{n+1})\cdots g_m(x_{n+m})$$
$$=f_1(x_1)\cdots f_n(x_n)\theta(x_1+\cdots+x_n)g_1(x_{n+1})\cdots g_m(x_{n+m})\varphi(x_1+\cdots+x_{n+m})$$
$$\in\mathcal{B}'(R^{(n+m)k}),$$

because :

$g_i\in\mathcal{B}'(R^k)$ by a) above,

$\varphi(x_1+\cdots+x_{n+m})\in\mathcal{B}(R^{(n+m)k})$ since it is bounded,

and

$$f_1(x_1)\cdots f_n(x_n)\theta(x_1+\cdots+x_n)\in\mathcal{B}'(R^{nk})$$

since $f_1*\cdots*f_n$ exists. Thus c) shows that the product of the f_i, θ, and the g_i belongs to $\mathcal{B}'(R^{(n+m)k})$, and b) allows us to include the function φ. This gives us the desired relation $(*)$. Q. E. D.

Now we obtain our main theorem of associativity.

THEOREM 2. *Let* $f_1,\cdots,f_n\in\mathcal{D}'(R^k)$ *be nonzero distributions such that the n-fold convolution* $f_1*\cdots*f_n$ *exists. Then for all* $m<n$, *the m-fold convolution* $f_1*\cdots*f_m$ *and the* $(n-m+1)$*-fold convolution* $(f_1*\cdots*f_m)*f_{m+1}*\cdots*f_n$ *exist and*

$$f_1*\cdots*f_n=(f_1*\cdots*f_m)*f_{m+1}*\cdots*f_n.$$

REMARK. By combining Theorem 2 with the commutative law above, it is easy to obtain any desired associativity relation, *provided* that the n-fold convolution exists. Thus, for example, it follows that

$$f_1*\cdots*f_n=(f_1*\cdots*f_m)*(f_{m+1}*\cdots*f_n).$$

The proof of Theorem 2 depends on the following lemmas.

LEMMA 2. (Existence of another mesa function). *For any $\varphi\in\mathcal{D}(R^k)$ and $\psi\in\mathcal{D}(R^{(n-m)k})$, $n>m$, there is a mesa function $\theta\in\mathcal{D}(R^k)$ such that*

$$\begin{aligned}&\varphi(x_1+\cdots+x_m)\psi(x_{m+1},\cdots,x_n)\\&\quad=\theta(x_1+\cdots+x_m+\cdots+x_n)\varphi(x_1+\cdots+x_m)\psi(x_{m+1},\cdots,x_n),\end{aligned}$$

where each $x_i\in R^k$.

Proof. Similar to that of Lemma 1.

Now we state two results from [1] which will be used in our proof, together with two related results of similar type.

LEMMA A (Fubini's theorem). *Let U and V be complementary subspaces of R^k, and let V be the sum of two complementary subspaces S, T. Suppose that $\int_V f(u,v)\,dv$ exists. Then the iterated integral $\int_S\int_T f(u,s,t)\,dt\,ds$ exists and equals $\int_V f(u,v)\,dv$.*

LEMMA B (Variable constants theorem). *Let $R^k=U\oplus V$, $V=S\oplus T$. Let $f(u)\in\mathcal{D}'(U)$, $f\neq0$, and $g(s,t)\in\mathcal{D}'(V)$. Then*

$$H(u,s)=\int_T f(u)g(s,t)\,dt$$

exists as a distribution on $U\oplus S$ if and only if

$$G(S)=\int_T g(s,t)\,dt$$

exists as a distribution on S, in which case

$$H(u,s)=f(u)G(s).$$

The proof of Lemmas A and B are given in [1, Theorem 2.16 and Theorem 2.17].

REMARK. The hypothesis that $f\neq0$ is only needed to prove $\int_T g(s,t)\,dt$ exists whenever $\int_T f(u)g(s,t)\,dt$ exists. If we assume that $\int_T g(s,t)\,dt$ exists, then Lemma B is true (and trivial) for $f=0$.

LEMMA C (Another variable constants theorem). *Let $R^k=U\oplus V$, and let*

$f(u,v)$ be a distribution on R^k such that the partial integral $\int_V f(u,v)\,dv$ exists. Let $k(u)$ be a C^∞ function. Then the partial integral $\int_V k(u)f(u,v)\,dv$ exists, and

$$\int_V k(u)f(u,v)\,dv=k(u)\int_V f(u,v)\,dv.$$

Proof. To show that $\int_V k(u)f(u,v)\,dv$ exists, let $\varphi(u)\in\mathcal{D}(U)$. Then since $k(u)$ is a C^∞ function, $\varphi(u)k(u)\in\mathcal{D}(U)$. Hence $\varphi(u)k(u)f(u,v)\in\mathcal{B}'(R^k)$ by the partial integrability of $f(u,v)$ over V.

For the equality, let us compute the following. Take $\varphi(u)\in\mathcal{D}(U)$. Then

$$\begin{aligned}&<\int_V k(u)f(u,v)\,dv,\ \varphi(u)>\\ &=\int_{R^k} k(u)\varphi(u)f(u,v)\,dv \qquad \text{(by Definition)}\\ &=<\int_V f(u,v)\,dv,\ k(u)\varphi(u)> \quad (k(u)\varphi(u)\in\mathcal{D}(U))\\ &=<k(u)\int_V f(u,v)\,dv,\ \varphi(u)> \quad (\text{since } \varphi(u)\in\mathcal{D}(U)).\end{aligned}$$ Q. E. D.

REMARK. In this case, unlike Lemma B, the product is not a tensor product. Hence the multiplier $k(u)$ is required to be C^∞. Furthermore it is not true that the existence of $\int_V k(u)f(u,v)\,dv$ implies that of $\int_V f(u,v)\,dv$, even if $k\neq 0$.

LEMMA D. *Let $f_1,\cdots,f_n\in\mathcal{D}'(R^k)$. Then the convolution $f_1*\cdots*f_n$ exists if and only if the following parial integral exists, and then*

$$\begin{aligned}&(f_1*\cdots*f_n)(x)\\ &=\int_{R^{(n-1)k}} f_1(x-x_2-x_3-\cdots-x_n)\ f_2(x_2)\cdots f_n(x_n)d(x_2,\cdots,x_n).\end{aligned}$$

Proof. $f_1*\cdots*f_n$ exists if and only if, for any $\varphi\in\mathcal{D}(R^k)$, $\varphi(x_1+\cdots+x_n)\quad f_1(x_1)\cdots f_n(x_n)\in\mathcal{B}'(R^{nk})$. Through a linear change of variables,

$$x=x_1+\cdots+x_n,\quad x_1=x-x_2-\cdots-x_n,$$
$$\varphi(x_1+\cdots+x_n)f_1(x_1)\cdots f_n(x_n)=\varphi(x)f_1(x-x_2-\cdots-x_n)f_2(x_2)\cdots f_n(x_n).$$

Hence $\varphi(x)f_1(x-x_2-\cdots-x_n)f_2(x_2)\cdots f_n(x_n)\in\mathcal{B}'(R^{nk})$ which is equivalent to $f_1(x-x_2-\cdots-x_n)f_2(x_2)\cdots f_n(x_n)$ being partially integrable over $R^{(n-1)k}$.

Equality follows from the definitions of convolution and partial integration. Q. E. D.

Proof of Theorem 2. For notational clarity, we will let $r=n-m$ and write the variables $x_{m+1}, \cdots, x_n$ as $y_1, \cdots, y_r$ and the functions $f_{m+1}, \cdots, f_n$ as $g_1, \cdots, g_r$.

Thus we assume that $f_1 * \cdots * f_m * g_1 * \cdots * g_r$ exists and that all f_i and $g_i \neq 0$. Let us first show that then $f_1 * \cdots * f_m$ is defined. By definition, $f_1 * \cdots * f_m$ is defined if and only if, for any $\varphi \in \mathcal{D}(R^k)$, $\varphi(x_1 + \cdots + x_m) f_1(x_1) \cdots f_m(x_m) \in \mathcal{B}'(R^{mk})$. Now we use the fact that the f_i and g_i are nonzero, and invoke the Variable constants theorem (Lemma B) : it follows that the condition

$$\varphi(x_1 + \cdots + x_m) f_1(x_1) \cdots f_m(x_m) \in \mathcal{B}'(R^{mk})$$

is equivalent to

$$\varphi(x_1 + \cdots + x_m) f_1(x_1) \cdots f_m(x_m) g_1(y_1) \cdots g_r(y_r)$$

being partially integrable over the subspace $R^{mk} \subseteq R^{nk}$.

To show partial integrability means to verify that

$$\psi(y_1, \cdots, y_r) \varphi(x_1 + \cdots + x_m) f_1(x_1) \cdots f_m(x_m) g_1(y_1) \cdots\cdots g_r(y_r) \in \mathcal{B}'(R^{nk})$$

for all $\psi \in \mathcal{D}(R^{rk})$. Following Existence lemma 2, choose a $\theta \in \mathcal{D}(R^k)$ so that

$$\begin{aligned}&\varphi(x_1 + \cdots + x_m) \psi(y_1, \cdots, y_r)\\ &= \theta(x_1 + \cdots + x_m + y_1 + \cdots + y_r) \varphi(x_1 + \cdots + x_m) \psi(y_1, \cdots, y_r).\end{aligned}$$

Then

$$\begin{aligned}&\psi(y_1, \cdots, y_r) \varphi(x_1 + \cdots + x_m) f_1(x_1) \cdots f_m(x_m) g_1(y_1) \cdots g_r(y_r)\\ &= \theta(x_1 + \cdots + x_m + \cdots + y_r) f_1(x_1) \cdots f_m(x_m) g_1(y_1) \cdots g_r(y_r) \cdots\\ &\qquad\qquad \psi(y_1, \cdots, y_r) \varphi(x_1 + \cdots + x_m)\end{aligned}$$

which belongs to $\mathcal{B}'(R^{nk})$ because of convolvability of $f_1, \cdots, f_m, g_1, \cdots, g_r$ [so that the product

$$\theta(x_1 + \cdots + x_m + y_1 + \cdots + y_r) f_1(x_1) \cdots f_m(x_m) g_1(y_1) \cdots g_r(y_r) \in \mathcal{B}'(R^{nk}),$$

and the product $\psi(\ )\varphi(\ ) \in \mathcal{B}(R^{nk})$]. Hence

$$\varphi(x_1 + \cdots + x_m) f_1(x_1) \cdots f_m(x_m) g_1(y_1) \cdots g_r(y_r)$$

is partially integrable over R^{mk} which, as we have seen, implies convolvability of $f_1, \cdots, f_m$.

This shows that the m-fold convolution $f_1 * \cdots * f_m$ exists. Now we must show that the $(r+1)$-fold convolution $(f_1 * \cdots * f_m) * g_1 * \cdots * g_r$ exists and equals $f_1 * \cdots * f_m * g_1 * \cdots * g_r$.

$$\begin{aligned}&\langle f_1 * \cdots f_m * g_1 * \cdots * g_r,\ \varphi \rangle\\ &= \int_{R^{nk}} \varphi(x_1 + \cdots + x_m + y_1 + \cdots + y_r) f_1(x_1) \cdots f_m(x_m) g_1(y_1) \cdots g_r(y_r)\\ &\qquad\qquad d(x_1, \cdots, x_m, y_1, \cdots, y_r).\end{aligned}$$

Now make a change of variables : $x = x_1 + \cdots + x_m$, the variables $x_2, \cdots, x_m, y_1, \cdots, y_r$ remaining as before; thus $x_1 = x - x_2 - \cdots - x_m$. Then we have

$$\langle f_1*\cdots*f_m*g_1*\cdots*g_r, \varphi\rangle$$
$$=\int_{R^{nk}}\varphi(x+y_1+\cdots+y_r)f_1(x-x_2-\cdots-x_m)f_2(x_2)\cdots f_m(x_m)$$
$$g_1(y_1)\cdots g_r(y_r)d(x, x_2, \cdots, x_m, y_1, \cdots, y_r)$$
$$=\int_{R^{(r+1)k}}\int_{R^{(m-1)k}}[\text{same}]\, d(x_2, \cdots, x_m)d(x, y_1, \cdots, y_r) \quad \text{(Fubini's theorem)}.$$

Now the C^∞ function $\varphi(x+y_1+\cdots y_r)$ can be moved forward by Lemma C, and then *after* that is done, the tensor product $g_1(y_1)\cdots g_r(y_r)$ can be moved forward using Lemma B (the variable constants theorem) to get

$$\int_{R^{(r+1)k}}\varphi(x+y_1+\cdots+y_r)g_1(y_1)\cdots g_r(y_r)\int_{R^{(m-1)k}}f_1(x-x_2=\cdots-x_m)f_2(x_2)$$
$$\cdots f_m(x_m)\, d(x_2, \cdots, x_m)d(x, y_1, \cdots, y_r)$$
$$=\int_{R^{(r+1)k}}\varphi(x+y_1+\cdots+y_r)g_1(y_1)\cdots g_r(y_r)\,(f_1*\cdots*f_m)(x)\, d(x, y_1, \cdots, y_r)$$
$$\text{(by Lemma D)}$$
$$=\langle(f_1*\cdots*f_m)*g_1*\cdots*g_r, \varphi\rangle. \qquad \text{Q. E. D.}$$

Having introduced a new definition of n-fold convolutions of distributions, we are obligated to see how it compares with other definitions. This comparison was carried out in [1, Section III] for 2-fold convolutions. Similar results hold here (with almost identical proofs). The classical definitions coincide with ours when both definitions apply. The new definition is more general, however, since it includes several classical cases.

Here we will verify that the new definition behaves properly with respect to the operation of taking partial derivatives.

LEMMA. *For any $f\in\mathcal{D}'(R^k)$, $f*\delta^{(\alpha)}$ is defined and equals $f^{(\alpha)}$, where α is a multi-index.*

Proof. Since $\delta^{(\alpha)}$ has compact support, the convolution $f*\delta^{(\alpha)}$ exists in the classical sense of distribution theory. Furthermore, it is well-known that $f*\delta^{(\alpha)}=f^{(\alpha)}$. The fact that our definition coincides with the classical one in the compact support case is proved in [1, Theorem 3. 1].

From the Lemma above and Theorems 1 and 2, we obtain the following:

COROLLARY. *If the convolution $f*g$ exists, then for any multi-index α, the convolutions $f^{(\alpha)}*g$ and $f*g^{(\alpha)}$ exist, and*

$$f^{(\alpha)}*g=(f*g)^{(\alpha)}. =f*g^{(\alpha)}.$$

A similar statement holds for n-fold convolutions.

COUNTEREXAMPLES. If we consider convolution as an iteration of binary operations, rather than an n-ary operation, then associativity does not hold.

Thus consider the law:

$$(f*g)*h=f*(g*h). \tag{A}$$

We give two examples. In the first, both sides of (A) exist, but they are not equal. In the second, one side of (A) exists and the other does not.

Example 1. Let

$$f(x)=1/2\ \mathrm{sign}(x)=\begin{cases}1/2 & \text{for } x>0\\ -1/2 & \text{for } x>0\end{cases}$$

$$g(x)=\delta'(x),\quad h(x)=1.$$

Then $(f*g)*h=h$, but $f*(g*h)=0$.

Example 2. Let

$$f(x)=\delta'(x),$$

$$g(x)=\begin{cases}1/x & \text{for } x>1\\ 0 & \text{for } x<1,\end{cases}$$

$$h(x)=1.$$

Then $(f*g)*h$ exists but $f*(g*h)$ does not: for $(f*g)=g'$ is the sum of a delta-function and an L^1 function, both of which can be convolved with 1; but $(g*h)$ does not exist since $g(x)\geq 0$ and $g(x)$ is not integrable.

Reference

1. H. Youn and I. Richards, *On the general definition of convolution for distributions*, Jour. of Korean Math. Soc. **17** (1980), 13-37.

[The paper [1] contains a more extensive list of references.]

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