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On some properties of ordinal spaces.

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1. Introduction

In this note we shall try to investigate the properties of ordinal spaces and by using these properties we will determine (1) whether some separation properties are inherited by each subspace topolgoy (2) whether some separation properties are transmitted to cartesian product and (3) some separation axioms imply others.

2. Ordinal spaces

Let Γ be any ordinal number and in $[0, \Gamma]$ we consider the topology $\mathcal{T}_0$ generated by all sets of the form $\{x|x>\alpha\}$ and $\{x|x<\beta\}$, where α, β are ordinal numbers such that $\alpha\leqslant\Gamma$, $\beta\leqslant\Gamma$, is called the ordinal topology and $[0, \Gamma]$ with this topology $\mathcal{T}_0$ is called the ordinal space.

From the definition of the ordinal space it can easily seen that the family $\{[\alpha, \beta]|\alpha<\beta\leqslant\Gamma\}$, where $[\alpha, \beta]=\{x|x>\alpha\}\cap\{x|x<\beta+1\}$, is a base for this topology $\mathcal{T}_0$ and that $[\alpha, \beta)$ is open if and only if $\alpha=0$ or if α has an immediate predecessor.

Theorem 1. *Every ordinal space* $[0, \Gamma]$ is compact.

Proof. Let $\mathfrak{A}$ be an open covering of $[0, \Gamma]$. And let $A=\{\alpha|[0, \alpha]$ is covered by finite subfamily of $\mathfrak{A}\}$, then A is bounded. Let B be the set of all upper bound of A, then $B\subset[0, \Gamma]$ and since $[0, \Gamma]$ is the well ordered set, B has the first element. Since $\beta\in[0, \Gamma]$ there is an open set $U\in\mathfrak{A}$ such that $\beta\in U$. Hence there is a basic open set (γ, δ) with $\gamma<\beta<\delta$. If there is β_1 such that $\gamma<\beta_1<\beta<\delta$, then $[0, \beta_1]$ is covered by a finite subfamily of $\mathfrak{A}$ and also $[0, \beta]$ is covered by the finite subfamily of $\mathfrak{A}$. If γ is the immediate predeessor of β then $[0, \gamma]$ is covered by a finite subfamily of $\mathfrak{A}$. Since $[0, \beta]=[0, \gamma]\cup\{\beta\}$, $[0, \beta]$ is also covered by a finite subfamily of $\mathfrak{A}$. If $\beta<\Gamma$ then $[0, \beta+1]$ is also covered by the finite members of $\mathfrak{A}$. This is contradiction. Hence we have $\beta=\Gamma$.

Corollary 1. *Every ordinal space is paracompact, hence also normal.*

In the following theorem we show that a subspace of a normal space need not be normal. Let Ω be the first uncountable ordinal, ω the th first infinite ordinal. Since $[0, \Omega]$ and $[0, \omega]$ are compact, $[0, \Omega]\times[0, \omega]$ is also compact, hence $[0, \Omega]\times[0, \omega]$ is in fact normal.

Theorem 2. *The subspace* $S=[0, \Omega]\times[0, \omega]-(\Omega, \omega)$ *of the normal space* $T=[0, \Omega]\times$

$[0, \omega]$ is not normal.

Proof. Let $A=\{(\Omega, n)|0\leqslant n<\omega\}$ and $B=\{(\zeta, \omega)|0\leqslant\zeta<\Omega\}$ then A and B are both closed and disjoint in S. We shall show that A and B can not be separated. Let U be any neighborhood of A. Since for each fixed n the point $(\Omega, n)\in U$, there is an ordinal $\alpha_n<\Omega$ such that $[\alpha_n, \Omega]\times\{n\}\subset U$. Since $\{\alpha_n|n\in Z^+\}$ is a countable subset of $[0, \Omega]$, it has an upper bound α_0 in $[0, \Omega]$, so that the tube $[\alpha_0, \Omega]\times[0, \omega)\subset U$. It follows that any neighborhood of (α_0+1, ω) in B must contain points of U. Therefore each neighborhood V of B will intersect U.

Theorom 3. *For each ordinal number Γ the ordinal space $[0, \Gamma]$ is completely normal.*

Proof. Let A and B be subsets of $[0, \Gamma]$ with $\bar{A}\cap B=\phi$ and $A\cap\bar{B}=\phi$. For each $\alpha\in A$ the set $\{\beta<\alpha|\beta\in B\}$ has a supremum b_α, which necessarily belongs to B. Note that $[b_\alpha, \alpha]$ is an open set containing no points of B. We thus get an open set $U=\cup\{[b_\alpha, \alpha]|\alpha\in A\}$ containing A, and similary, an open set $V=\cup\{[a_\beta, \beta]|\beta\in B\}$ containing B. Now U and V are disjoint. For, if $U\cap V\neq\phi$, then some $[b_\alpha, \alpha]\cap[a_\beta, \beta]\neq\phi$. Assuming, say, that $\beta<\alpha$, this gives $\beta\in[b_\alpha, \alpha]$, which is impossible.

In theorem 2, A and B are the sets with $\bar{A}\cap B=\phi$, $A\cap\bar{B}=\phi$ in the product space $T=[0, \Omega]\times[0, \omega]$. On the other hand the proof of theorem 2 shows that T is not completely normal. Thus we have the following theorem:

Theorem 4. *The cartesian product of completely normal spaces need not be completely normal.*

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