韓國數學敎育學會誌 <數學敎育>
1981. 6 Vol. XIX. No. 2

Journal of the Korea Society of Mathematical Education
Jun. 1981. Vol. XIX No. 2

Uniform Convergence on Compact Spaces

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I. Introduction

A uniform convergence of sequence of functions f_n on compact spaces plays an important role in analysis. W.C. Waterhouse [4] proved the following theorem:

Theorem. Let X and Y be compact spaces. Let $f_1, f_2, \cdots\cdots$ be functions from X to Y, with graphs $\Gamma(f_1)$, $\Gamma(f_2), \cdots\cdots$ in $X \times Y$.

Then Lim $\Gamma(f_n)$ exists and is the graph of a function f if and only if f_n converges uniformly to f and f is continuous.

In this note we give the alternative proof of the above result and we prove a uniform convergence of equicontinuous sequence $\langle f_n \rangle$ on compact space by means of graph convergence.

Let $\langle S_n \rangle$ be a sequence of sets in some metric space X.

We define Lim sup (S_n) to be the set of points that S_n repeatedly approaches, i.e., the q in X such that every neighborhood of q meets infinitely many S_n. Similarly we let Lim inf (S_n) be the set of points that S_n ultimately stays close to, i.e., the q in X such that every neighborhood of q meets all but finitely many S_n. If these two sets concide, we define Lim (S_n) to be their common value. Next we make some preliminary observations about these concepts. First, obviously Lim $\inf(S_n) \subset$ Lim $\sup(S_n)$. Second, the definitions can be paraphrased as follows: a point q is in Lim $\inf(S_n)$ iff there is a sequence of points P_n in S_n converging to q and q is in Lim $\sup(S_n)$ iff there is a subsequence S_{n_i} with points p_i in S_{n_i} converging to q. Finally, Lim $\inf(S_n)$ and Lim $\sup(S_n)$ are always closed, so that Lim $\inf(S_n)$ is closed when it exists.

Indeed, take x in the closure of Lim $\inf(S_n)$. Given $\varepsilon > 0$, we can find q in Lim $\inf(S_n)$ with $d(x, q) < \varepsilon/2$. For all sufficiently large m there is a point p_m in S_n with $d(q, p_m) < \varepsilon/2$, and then $d(x, p_m) < \varepsilon$; thus x is in Lim $\inf(S_n)$.

II. Main theorems

Theorem 1. *Let X and Y be compact metric spaces. Let f_1, $f_2 \cdots\cdots$ be functions from X to Y, with graphs $\Gamma(f_1)$, $\Gamma(f_2), \cdots\cdots$ in $X \times Y$. Then Lim $\Gamma(f_n)$ exists and is the graph of a function f if and only if f converges uniformly to f and f is continuous.*

Proof. First, we show that f is continuous as $\Gamma(f)=\text{Lim}\ \Gamma(f_n)$, $\Gamma(f)$ is a closed set and a subset of the compact space $X\times Y$.

Let $<(x_n, f(x_n))>$ be a sequence in $\Gamma(f)$ such that sequence x_n converges to x. By compactness of $\Gamma(f)$, there exists a subsequence of a sequence $<(x_n, f(x_n))>$ which converge to a $(x,y)\in\Gamma(f)$, and so $y=f(x)$. Hence f is continuous. Next, we show that f_n uniformly converges to f. If not, then for some $\varepsilon>0$ we have a subsequence f_{n_i} and x_i such that $d(f_{n_i}(x_n), f(x_n))\geq\varepsilon$. By compactness of $X\times Y$, we have a convergent subsequence $<(x_{k_i}, f_{n_{k_i}}(x_{k_i}))>$ of a sequence $<(x_i, f_{n_i}(x_i))>$. If $(x_k, f_{n_{k_i}}(x_{k_i}))$ converges to (x, y), then (x,y) belongs to $\Gamma(f)$, and so $y=f(x)$. But as f is continuous, $\lim_{x_{k_i}\to x} f(x_{k_i})=f(x)$. Hence $d(f_{n_{k_i}}(x_{k_i}), f(x_{k_i}))\longrightarrow 0$, which is absurd because they are supposedly at distance at least ε from each other.

Conversely let f_n be a uniformly converges to a continuous function f as $\lim_{n\to\infty}(x, f_n(x))=(x,f(x))$, $\Gamma(f)$ is a subset of Lim inf $\Gamma(f_n)$, and we must show only Lim sup $\Gamma(f_n)\subset\Gamma(f)$.

If (x,y) is in the Lim sup (f_n), there is a sequence $<(x_i, f_{n_i}(x_i))>$ converging to (x, y), where $<(x_i, f_{n_i}(x_i))>$ is in $\Gamma(f_{n_i})$. But as $f_{n_i}\longrightarrow f$ uniformly, $\lim_{n\to\infty} x_i=x$ and $d(f_{n_i}(x_i), f(x))\leq d(f_{n_i}(x_i), f(x_i))+d(f(x_i), f(x))\longrightarrow 0$, $(x,y)=(x, f(x))\in\Gamma(f)$. Hence Lim sup $\Gamma(f_n)\subset\Gamma(f)$. Therefore we completes above the proof.

Theorem 2. *Let X and Y be compact metric spaces. Let $<f_n>$ be an equicontinuous sequence of functions to Y which converges pointwise to a function f. Then $<f_n>$ converges uniformly to f.*

Proof. If (x,y) is in the $\Gamma(f)$, $(x,y)=(x, f(x))=\lim_{n\to\infty}(x, f_n(x))$. Hence we have $\Gamma(f)\subseteq$ Lim inf $\Gamma(f)$. Next we show that Lim sup $\Gamma(f_n)$ is contained in $\Gamma(f)$. If (x,y) is in Lim sup $\Gamma(f_n)$, there is a sequence $<(x_i, f_{n_i}(x'))>$ converging to it.

Choose $\varepsilon>0$. By equicontinuity, there is a neighborhood $N(x)$ of x such that $d(f_{n_i}(x_i), f_n(x))<\varepsilon/2$ for all x_i in $N(x)$ and all n. Choose N_1 so large that for all $n\geq N_1$, we have $d(f_n), f(x))<\varepsilon/2$ and since $\lim x_i=x$, we may choose N_2 so large that x_i is in $N(x)$ for all $n_i\geq N_2$. Hence let N be Max $\{N_1, N_2\}$.

For all $n_i\geq N$, we have

$$d(f_{n_i}(x_i), f(x))\leq d(f_{n_i}(x_i), f_{n_i}(x))+d(f_{n_i}(x), f(x))<\varepsilon/2+\varepsilon/2=\varepsilon$$

Hence $<f_{n_i}(x_i)>$ converge to $f(x)$. Since $(x,y)=(x,f(x))$, we have Lim sup $\Gamma(f_n)\subset\Gamma(f)$. It is easy from theorem 1. to see that $<f_n>$ converges uniformly to f.

References

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