韓國數學敎育學會誌 <數學敎育>
1981. 6. Vol. XIX. No. 2

Journal of the Korea Society of Mathematical Education
Jun. 1981. Vol. XIX. Nc. 2

On Some Compatible Matrix Norms

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1. Definitions.

(i) Let M_n denote the set of all $(n\times n)$ matrices. A *matrix norm* for M_n is a real-valued function $\|\cdot\|$ defined on M_n satisfying for all $A, B\in M_n$:

(a) $\|A\|\geq 0$ and $\|A\|=0$ iff $A=0$, (b) $\|\alpha A\|=|\alpha|\,\|A\|$ for any scalar α,

(c) $\|A+B\|\leq\|A\|+\|B\|$, (d) $\|AB\|\leq\|A\|\,\|B\|$.

(ii) A matrix norm is said to be *consistent* or *compatible* with the vector norm $\|\cdot\|_V$ iff

$$\|A\mathbf{x}\|_V\leq\|A\|\,\|\mathbf{x}\|_V \text{ for all } \mathbf{x}\in V \text{ and for all } A\in C^{n^2}. \tag{1}$$

(iii) Let A be an arbitrary matrix. The *spectrum* of A is the set of all eigenvalues of A, and it is denoted by $\delta(A)$. The *spectral radius* is the maximum size of these eigenvalues, and it is denoted by $\rho(A)=\max_{\lambda\in\delta(A)}|\lambda|$.

2. Compatible matrix norms.

(i) The operater norm.

For a matrix A, the *operator norm* induced by $\|\mathbf{x}\|_p$ is denoted by $\|A\|_p$.

The operator norm defined by

$$\|A\|=\|A\|_V=\sup_{\mathbf{x}\neq 0}\frac{\|A\mathbf{x}\|_V}{\|\mathbf{x}\|_V},\quad \mathbf{x}\in C^n \tag{2}$$

satisfies the conditions (a) and (b). And by its definition, it satisfies the property (1) From this it can be shown that (2) satisfies the conditions (c) and (d). Hence the operator norm defined by (2) is a compatible matrix norm. Especially the norm $\|A\|=\sup_{\|\mathbf{x}\|=1}\|A\mathbf{x}\|$ is called a *subordinate* or *induced norm.* And the most common operator norms are the *maximum (absolute) column norm* and *row norm* respectively:

$$\|A\|_1=\max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}|,\quad \|A\|_\infty=\max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}|.$$

(ii) The Euclidean matrix norm.

$$\|A\mathbf{x}\|_2=\Big[\sum_{i=1}^{n}\Big|\sum_{j=1}^{n}a_{ij}x_j\Big|^2\Big]^{1/2}\leq\Big[\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|a_{ij}|^2\Big)\Big(\sum_{j=1}^{n}|x_j|^2\Big)\Big]^{1/2}$$

$$\leq F(A)\,\|\mathbf{x}\|_2,\quad F(A)=\Big[\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\Big]^{1/2} \tag{3}$$

$F(A)$ is called the *Frobenious norm* of A. The conditions (a) through (d) are satisfied since $F(A)$ is just the Euclidean norm on C^{n^2}. Thus $F(A)$ is a matrix norm, compatible with the Euclidean vector norm. So we denote $F(A)$ by $\| A \|_e$. And $F(A)$ is not subordinate since $F(I)=\sqrt{n}$.

(iii) The Hilbert or spectral norm.

Now let $g(A)\equiv L_2=\min\{L : \| A\mathbf{x} \|_2 \le L \| \mathbf{x} \|_2, \text{ for all } \mathbf{x}\in R^n\}$.

Then $g(A)$ satisfies the all conditions (a) through (d) of a matrix norm. Thus we write $g(A)\equiv \| A \|_2$. From above definition it fellows that $g(A)\equiv \| A \|_2 \le \| A \|_e$, for all $A\in M_n$, and that $\| A \|_2$ is compatible with the the $\| \cdot \|_2$ vector norm.

3. Theorem $\| A \|_3=\sqrt{\rho(A^*A)}$.

Proof: A^*A is Hermitian and its eigenvalues $\{\lambda_i\}_{i=1}^n$ are real and nonnegative.

Let $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n\ge 0$. The eigenvectors V_i of A^*A can be chosen to be orthonomal, $\mathbf{v}_i^*\mathbf{v}_j=\delta_{ij}$, $1\le i,j\le n$. Then any vector $\mathbf{x}\neq\mathbf{0}$ can be written as $\mathbf{x}=\sum_{i=1}^n a_i\mathbf{v}_i$.

Thus we can verify that

$$\left(\frac{\| A\mathbf{x} \|_2}{\| \mathbf{x} \|_2}\right)^2 \equiv \frac{\mathbf{x}^*A^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \frac{\sum_{i=1}^n |a_i|^2\lambda_i}{\sum_{i=1}^n |a_i|^2} \tag{4}$$

which implies $0\le\lambda_n\le\left(\frac{\| A\mathbf{x} \|_2}{\| \mathbf{x} \|_2}\right)^2\le\lambda_1$, or $\| A\mathbf{x} \|_2\le\sqrt{\lambda_1}\,\| \mathbf{x} \|_2$ (5)

for all vectors $\mathbf{x}$. Thus $\sqrt{\lambda_1}=\rho(A^*A)\ge L_2=\| A \|_2$. Now substitute $\mathbf{v}_1$ for $\mathbf{x}$ in (4) and (5) becomes $\| A\mathbf{v}_1 \|_2=\sqrt{\lambda_1}\,\| \mathbf{v}_1 \|_2$. This implies that $\sqrt{\rho(A^*A)}=L_2$.

4. Conclusion.

Which is the most desirable matrix norm? There are certainly differences in their cost, e. g., some will require more expensive computation than others. The spectral norm is usualy the most expensive. The answer to the question depends in part on the use for the norm. In most instance, we want the norm that puts the smallest upper bound on the magnitude of the matrix. In this sense, the spectral norm is best.

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