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On Some Properties of Ordinal Space

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Abstract: In this paper, we give several properties of ordinal spaces $[0, \Omega]$ and $[0, \Omega[$.

1. Introduction.

Let Γ be any ordinal number and let $[0, \Gamma]$ be a space with the topology generated by all sets of form $\{x|x>\alpha\}$ and $\{x|x<\beta\}$. We call this topological space the ordinal space $[0, \Gamma]$. Throughout, Ω denotes the first uncountable ordinal number, and we mainly investigate the properties of the space $[0, \Omega]$ and its subspace $[0, \Omega[$.

2. Some Properties of Ordinal Space.

Lemma 1. *Every nonincreasing sequence of ordinal numbers is necessarily finite.* (*See* [1], p. 43)

Lemma 2. *Let* $f: [0, \Omega[\longrightarrow[0, \Omega[$ *be any map such that* $f(\alpha)<\alpha$ *for all* $\alpha \geqslant$ *some* α_0. *Then there exists a* $\beta_0<\Omega$ *with the following property: As* α *increases, its image* $f(\alpha)$ *repeatedly returns to value below* β_0. *In symbols:* $\exists \beta_0\ \forall \beta\ \exists \alpha \geqslant \beta: f(\alpha) \leqslant \beta_0$. (See [1], p. 55)

We recall that a Hausdorff space X is paracompact if each open covering of X has an open nbd-finite refinement.

Proposition 1. *The ordinal space* $[0, \Omega]$ *is paracompact.*

Proof Let $\{U_\alpha|\alpha\in A\}$ be any open covering. Since the sets $]\lambda, \mu]$ form a basis, define $\varphi: [0, \Omega]\longrightarrow[0, \Omega]$ by associating with each $\beta \neq 0$ a $\varphi(\beta)<\beta$ such that $]\varphi(\beta), \beta]\subset U_\alpha$ for some α, and setting $\varphi(0)=0$. By induction, construct a sequence $\beta_0=\Omega$, $\beta_1=\varphi(\Omega), \cdots, \beta_n=\varphi(\beta_{n-1}), \cdots$; then $\beta_0>\beta_1>\cdots$. By Lemma 1, this terminates with some β_n. Because the process can not be continued, $\beta_n=0$, and so $]0, \Omega]\subset \bigcup_{i=1}^{n}]\beta_i, \beta_{i-1}]$. Choosing a U_{α_i} containing each $]\beta_i, \beta_{i-1}]$ and some $U_{\alpha_0}\supset\{0\}$, we have a finite subcovering of $\{U_\alpha|\alpha\in A\}$, which is consequently an open nbd-finite refinement.

Proposition 2. *The ordinal space* $[0, \Omega[$ *is not paracompact.*

Proof The open covering by the sets $[0, \alpha[$, $0<\alpha<\Omega$, has no open nbd-finite refinement. For, given any open refinement $\{U_\alpha\}$. define $\varphi: [0, \Omega[\longrightarrow[0, \Omega[$ as in proposition 1. Because of Lemma 2, there must be some β_0 such that $\forall \gamma\ \exists \beta>\gamma: \varphi(\beta)\leqslant\beta_0$, and it follows easily that β_0+1 is contained in infinitely many sets U_α.

Proposition 3. *Let Γ be any ordinal number. Then the ordinal space $[0, \Gamma[$ is normal. In particular, $[0, \Omega[$ is normal.*

Proof Let A and B be disjoint closed sets. For each $\alpha \in A$, the set $\{\beta<\alpha \mid \beta \in B\}$ has a supremum b_α (Lemma 1), which necessarily belongs to $\bar{B}=B$; note that $]b_\alpha, \alpha]$ is an open set containing no points of B. We thus get an open set $U=\cup\{]b_\alpha, \alpha] \mid \alpha \in A\} \supset A$, and similarly, an open $V=\cup\{]a_\beta, \beta] \mid \beta \in B\} \supset B$. Now assuming $U \cap V \neq \phi$, then some $]b_\alpha, \alpha] \cap]a_\beta, \beta] \neq \phi$. Since $\beta<\alpha$, this gives $\beta \in]b_\alpha, \alpha]$, which is impossible. Thus, $U \cap V=\phi$. That is, $[0, \Gamma[$ is normal.

A normal topological space in which each closed set is a G_δ is called perfectly normal. Then we have the following lemma.

Lemma 3. *The space $[0, \Omega]$ is not perfectly normal.*

Proof The singleton set $\{\Omega\}$ is a closed set in $[0, \Omega]$, but $\{\Omega\}$ is not G_δ-set. For, if $\{G_i \mid i \in N\}$ is arbitrary countable collection of open sets containing Ω, then because the sets $]\alpha, \beta]$ are a basis, $\forall i \ \exists\ \alpha_i<\Omega :]\alpha_i, \ \Omega] \subset G_i$. Being countable, the collection $\{\alpha_i \mid i \in N\}$ has an upper bound $\beta<\Omega$, so $\bigcap_{i=1}^{n} G_i \supset \ ]\beta, \Omega] \neq \{\Omega\}$. Hence, the space $[0, \Omega]$ is not perfectly normal.

Lemma 4. *Every metric space is perfectly normal.* (See [1], p. 186)

Lemma 5. (A. H. Stone) *Every metric space is paracompact.* (See [1], p. 186)

By the above three Lemmas, we can derive the following properties. Their proofs are clear.

Proposition 4. *The space $[0, \Omega]$ is not metrizable.*

Porposition 5. *The space $[0, \Omega[$ is not metrizable.*

In the proof of Proposition 1, we have the following result.

Proposition 6. *The space $[0, \Omega]$ is compact.*

Remark: We know that the usual topology on the real line R is metrizable, but not compact. And in Proposition 4 and Proposition 6, the space $[0, \Omega]$ is not metrizable, but compact. So we can say that compactness and metrizability are not related.

3. Application for perfect map.

Definition. A map $p : X \longrightarrow Y$ is called *perfect* if it is a continuous closed surjection and each fiber $p^{-1}(y)$ is compact.

Perfect maps preserve certain properties under inverse images. We know the following theorem.

Theorem. *Let $p : X \longrightarrow Y$ be a perfect map. Then*

(1) *If Y is paracompact, so also is X.*

(2) *If Y is compact, so also is X.*

(3) *If Y is Lindelöf, so also is X.*

(4) *If Y is countably compact, so also is X.*

In the above condition, if Y is normal, then is X normal? The answer is "No". In the below, we will give the counter example.

Lemma 6. *Let X be arbitrary, Y be Hausdorff, and $f: X \longrightarrow Y$ be continuous. Then the graph of f is closed in $X \times Y$. (See [1], p. 140)*

Lemma 7. *The set $\{(\alpha, \alpha) | 0 \leq \alpha < \Omega\}$ is closed in $[0, \Omega] \times [0, \Omega[$.*

Proof Let $\varphi: [0, \Omega[\longrightarrow [0, \Omega]$ be a map such that $\varphi(\alpha) = \alpha$ for all $0 \leq \alpha < \Omega$. Since $]\alpha, \beta]$ is a basis in $[0, \Omega]$, we must show: $\varphi^{-1}(]\alpha, \beta])$ is open. If $\beta \neq \Omega$, $\varphi^{-1}(]\alpha, \beta]) =]\alpha, \beta]$, if $\beta = \Omega$, $\varphi^{-1}(]\alpha, \Omega]) =]\alpha, \Omega[$. In any case, $\varphi^{-1}(]\alpha, \beta])$ is open in $[0, \Omega[$, so φ is a continuous map. By Lemma 6, the graph of φ is closed in $[0, \Omega[\times [0, \Omega]$, that is, $\{(\alpha, \alpha) | 0 \leq \alpha < \Omega\}$ is closed in $[0, \Omega[\times [0, \Omega]$. Thus $\{(\alpha, \alpha) | 0 \leq \alpha < \Omega\}$ is closed in $[0, \Omega] \times [0, \Omega[$.

Proposition 7. *The space $[0, \Omega] \times [0, \Omega[$ is not normal.*

Proof Let $A = \{(\Omega, n) | 0 \leq n < \Omega\}$ and $B = \{(\alpha, \alpha) | 0 \leq \alpha < \Omega\}$. Then A and B do not intersect closed sets in $[0, \Omega] \times [0, \Omega[$. Let U be any nbd of A; since for each fixed n the point $(\Omega, n) \in U$, there exists an ordinal $\alpha_n < \Omega$ such that $]\alpha_n, \Omega] \times \{n\} \subset U$. Then, since $\{\alpha_n | 0 \leq n < \Omega\}$ is a countable collection, $\{\alpha_n\}$ has an upper bound $\alpha_0 < \Omega$ so that the "tube" $]\alpha_0, \Omega] \times [0, \Omega[\subset U$. It follows that any nbd of $(\alpha_0 + 1, \alpha_0 + 1) \in B$ must contain points of U; therefore each $V \supset B$ will intersect U. Thus $[0, \Omega] \times [0, \Omega[$ is not normal.

Claim: *If $p: X \longrightarrow Y$ is perfect and Y is normal, then X need not be normal.*

Proof Let $p: [0, \Omega] \times [0, \Omega[\longrightarrow [0, \Omega[$ be the projection. Then p is a obviously a perfect map. The space $[0, \Omega[$ is normal, but $[0, \Omega] \times [0, \Omega[$ is not normal.

Reference:

[1] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1970.