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The Existence of A C^m Function Under Borel's Theorem

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1. Introduction

Let N be the set of all nonnegative integers.

We call an element p of N^n a "multi-index", and the "order" $|p|$ of a multi-index $p=(p_1, \cdots, p_n)\in N^n$ is defined by $|p|=p_1+\cdots+p_n$.

For any $p=(p_1, \cdots, p_n)\in N^n$, we shall use the notation

$$\left(\frac{\partial}{\partial X}\right)^p=\frac{\partial^{|p|}}{\partial X_1^{p_1}\cdots\partial X_n^{p_n}} \text{ or } \partial^p$$

Let Ω be an open subset of R^n, n-dimensional Euclidean space and let $(K_j)_{j=1,2,\ldots}$ be an increasing sequence of compact subsets of Ω whose union is Ω. Denote by $C^m(\Omega)$ $(0\le m\le\infty)$ the space of all complex valued functions having continuous partial derivatives of order $\le m$ and $C_c^\infty(\Omega)$ the space of C^∞ complex valued functions in Ω having compact support.

For every $j=1, 2, \cdots$ define the seminorm

$$P_{m,j}(\varphi)=\sup_{X\in K_j, |p|\le m}|\partial^p\varphi(X)|, \quad \forall\varphi\in C^m(\Omega).$$

The family of seminorms $(P_{m,j})_{j=1,2,\ldots}$ defines a Hausdorff locally convex topology on $C^m(\Omega)$. Since it is a countable family, every element of $C^m(\Omega)$ has a countable basis of neighborhoods, hence $C^m(\Omega)$ is a metrizable space. We know that the space $C^m(\Omega)$ with this topology is a Frechet space.

Let us denote by $C[[X]]$ the vector space of formal power series in n indeterminates $X_1, \ldots, X_n$, with complex coefficients, that is to say the series

$$u=\sum u_pX^p,$$

where the summation is performed over all the vectors $p=(p_1, \ldots, p_n)$. We can view u as a sequence depending on n indices, $p_1, \ldots, p_n$, $\{u_p\}$, with no condition whatever on the complex number which constitute it. If we provide $C[[X]]$ with the topology defined by the seminorms

$$|u|_m=\sup_{|p|\le m}|u_p|, \quad m=0, 1, \ldots,$$

then $C[[X]]$ is a locally convex metrizable space. We also know that $C[[X]]$ is a Frechet space.

2. Preliminaries

Definition. A continuous linear functional on $C_c^\infty(\Omega)$ is called a *distribution* in Ω and the *dual* of $C_c^\infty(\Omega)$ is denoted by $\mathcal{D}'(\Omega)$. Thus $\mathcal{D}'(\Omega)$ is the space of all distributions in Ω.

We see that functions of $C^m(\Omega)$, $0 \le m \le \infty$, L^p functions, $1 \le p \le \infty$, define distributions on Ω.

The Dirac measure δ on R^n defined by

$$\delta(\phi) = \phi(0), \quad \forall \phi \in C_c^\infty(R^n)$$

is a distribution. We know that the Dirac measure has compact support equal to $\{0\}$.

We shall prove the classical theorem of E. Borel to make this article self contained and to use the results in next section.

We shall use the following lemmas without proof which can be found in [4].

Lemma 2.1. *The distributions in R^n which have their support at the origin are the finite linear combinations of the derivatives of the Dirac measure at* 0.

Lemma 2.2. *Let E, F be two Frechet spaces and E', F' be their dual spaces respectively. Let u be a continuous linear map of E into F. Then u maps E onto F if and only if the following two conditions are satisfied:*

(*a*) *the transpose of u, ${}^t u : F' \to E'$, is injective;*

(*b*) *the image of t_u, ${}^t u(F')$, is weakly closed in E'.*

Theorem 2.3. *Let Φ be an arbitrary formal power series in n indeterminates, with complex coefficients. There is a C^∞ function φ in R^n whose Taylor expansion at the origin is indentical to Φ.*

In other words if, for every n-tuple $p = (p_1, \ldots, p_n)$ of integers $p_j \ge 0$, we give ourselves arbitrarily some complex number a_p, there is a C^∞ function φ in R^n such that $(\partial/\partial x)^p \varphi|_{x=0} = a_p$ for every p. Of course, the origin in R^n can be replaced by any other point.

Proof. Let us denote by u the mapping which assigns to every function $\varphi \in C^\infty(R^n)$ its Taylor expansion at the origin. We regard the latter as an element of the spaces Q_n of formal power series in n letters with complex coefficients. We must show that u is a surjection. We provide $C^\infty(R^n)$ with the natural C^∞ topology and Q_n with the topology of simple convergence of the coefficients. The dual of $C^\infty(R^n)$ is the space $\mathcal{E}'$ of distributions with compact support in R^n; the dual of Q_n is the space $\mathcal{P}_n$ of polynomials in n letters with complex coefficients. Observe that the mapping u is the mapping

$$\varphi \to \sum_{p \in N^n} \frac{1}{p!} [(\partial/\partial x)^p \varphi(0)] X^p.$$

If $\langle , \rangle$ denotes the bracket of the duality between $C^\infty(R^n)$ and $\mathcal{E}'$ on one hand, and between $\mathcal{P}_n$ and Q_n on the other, we see that we have, for any polynomial

$$P(X) = \sum_{p \in N^n} P_p X^p,$$

$$\langle P, u(\varphi) \rangle = \sum_{p \in N^n} \frac{1}{p!} P_p [(\partial/\partial x)^p \varphi(0)] = \langle \bar{P}(-\partial/\partial x)\delta, \varphi \rangle$$
$$= (-1)^{|p|} \langle \delta, \ \bar{P}(-\partial/\partial x)\varphi \rangle = (-1)^{|p|} \bar{P}(-\partial/\partial x)\varphi(0),$$

where δ is the Dirac measure at the origin and where we have set

$$\bar{P}(-\partial/\partial x)=\sum_{P\in N^n}(-1)^{|p|}\frac{1}{p!}P_p(\partial/\partial x)^p.$$

This means that the transpose ${}^t u$ of u is the mapping $P\mapsto\bar{P}(-\partial/\partial x)\delta$ of $\mathcal{P}_n$ into $\mathcal{E}'$. It is clear that the image of ${}^t u$ is the space of all linear combinations of derivatives of the Dirac measure at 0. By Lemma 2.1., this space is identical to the space of distributions having the origin $\{0\}$ as support and trivially closed, i.e., weakly closed. If we apply for instance the Fourier transformation to $\bar{P}(-\partial/\partial x)\delta$, then we know that ${}^t u$ is one-to-one. Hence, by Lemma 2.2., the proof is complete.

3. Main results

We donote by supp f the support of the function f, i.e., the closure of the set of points at which f does not vanish.

Theorem 3.1. *Let (P_i) be a suquence in R^n which has no limit point. Let Φ_i be an arbitrary formal power series in n indeterminates, with complex coefficients. Then there is a C^∞ function φ in R^n whose Taylor expansion at each point P_i is identical to Φ_i respectively.*

Proof. By Theorem 2.3., for each $i=0, 1, \ldots$, there exists a C^∞ function φ_i in R^n whose Taylor expansion at point P_i is identical to Φ_i. Let χ_i be a cutoff function, satisfying supp $\chi_i\cap$ supp $\chi_j=\phi$ for $i\neq j$, which has a value 1 in a sufficiently small neighborhood Q_i of P_i and vanishes outside of $2Q_i$. If we set $\varphi=\sum_{n=0}^{\infty}\chi_i\varphi_i$, then we have proved.

Theorem 3.2. *Let (P_i) be a sequence of points in R^1 which has a limit point 0. Let (a_i) be a sequence which converges to 0. Then there is a function $\varphi\in C^\infty(R^1-0)$* such that *$\varphi(P_i)=a_i$ and φ is a continuous function in R^1.*

Of course, the origin in the real line can be replaced by any other point.

Proof. Without loss of generality, we prove the case that (P_i) is a monotone sequence of points. For each $i=0, 1, \ldots$, let φ_i be a C^∞ function in R^1 such that $\varphi_i(P_i)=a_i$. Let $\chi_i(t)$ be a C^∞ function on the real line, for each ε_i depending on P_i, vanishing for $|t-P_i|\geq 2\varepsilon_i$ and equal to 1 for $|t-P_i|\leq\varepsilon_i$ such that supp $\chi_i\cap$ supp $\chi_j=\phi$ for $i\neq j$. Let $\varphi=\sum_{i=0}^{\infty}\chi_i\varphi_i$ and $\varphi(0)=0$. Consider an open interval U_i containing P_i such that supp $\chi_i\subset U_i$, $\sup_{t\in U_i}|\varphi(t)|\leq 2|a_i|$ and $U_i\cap U_j=\phi$ for $i\neq j$. To complete the proof, it suffices to show that φ is continuous at 0. Given $\varepsilon>0$, there is a positive integer m such that $j>$m implies $|a_j|<\varepsilon/2$. Then we have, for t with $|t|<|P_N|$, $\varphi(t)=0$ or $t\in$supp $\chi_i\varphi_i$ for some $i>N>m$. For the first case, there is no problem. In the second case, we get, for some $i>N>m$,

$$|\varphi(t)|\leq\sup_{s\in U_i}|\varphi(s)|\leq 2|a_i|<\varepsilon$$

If we take $\delta=|P_N|$, then $|t|<\delta$ implies $|\varphi(t)|<\varepsilon$. Since ε is arbitrary, φ is continuous at 0.

Corollary 3.3. *Let (P_i) be a sequence of points in R^n which has a limit point P. Let $(a_{ik})_{k=0}^{\infty}$ be a convergent sequence. Then there is a function $\varphi\in C^\infty(R^n-P)$ such that $\partial^k\varphi$*

$(P_i)=a_{ik}$, *for* $i, k\geq 0$ *and* φ *is a continuous function in* R^n.

Proof. We consider for convenience that (P_i) is a monotone sequence. Let the sequence $(a_{ik})_{i=0}^{\infty}$ converge to b_k as P_i converges to P. Define a function f at P by $f(X)=\sum_{j=0}^{\infty}\frac{b_j}{j!}(X-P)^j$ which has a sufficiently large radius of convergence r. Then $\{a_{ik}-\partial^k f(P_i)\}_{k=0}^{\infty}$ is a sequence which converges to 0 in $|X-P|<r$. By Theorem 3.2., there is a C^∞ function h in R^n-P and continuous on R^n. Let ξ be a C^∞ function on R^n which is equal to 1 in $|X-P|<r-\varepsilon$, for $\varepsilon>0$ and vanishing in $|X-P|\geq r$. If we set $\varphi(X)=h(X)+\xi(X)f(X)$, The proof has been done.

Remark. Let us remark that under the above conditions, φ is not a C^m function in R^n for $m\geq 1$. The following example shows this fact.

Example. Let $(P_n)=\left(\frac{1}{n+1}\right)$ be a sequence of points in the real line. If we put $a_{no}=\frac{1}{n+1}$ and $a_{nk}=0$ for $k\geq 1$, then we have a function $\varphi\in C^\infty(R^1-0)$ and $\varphi\in C^0(R^1)$. However φ is not a C^1 function in the real line. In fact, the sequence $\{\varphi'(P_n)\}=(a_{n1})$ converges to 0 as $P_n\to 0$. But, for any small neighborhood of the origin, there is a point t_n in an open interval (P_n, P_{n+1}) such that $\varphi'(t_n)=1$, by the Mean Value Theorem.

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