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—Research Paper—

Net Inventory Positions in ⟨Q, r⟩ Systems with Non-Stationary Poisson Demand Processes

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Abstract

In both continuous-review and periodic-review non-stationary inventory systems, the non-stationary Poisson demand process and the associated inventory position processes were proved being mutually independent of each other, which lead to the probability distribution of the corresponding net inventory position process in the form of a finite product sum of those two process distributions. It is also discussed how these results can correspond to analytical stochastic inventory cost function formulations in terms of the probability distributions of the processes.

1. Introduction

Inventory systems are operated largely based on some operating policies concerning review systems and ordering rules. The so-called transactions-reporting (continuous-review) systems and periodic-review systems are commonly used for inventory system review.

In both inventory systems the inventory position $\{IP_t;\ t\geq 0\}$ totally depends upon the demand process $\{N_t;\ t\geq 0\}$. Therefore, once it is verified that $\{IP_{t-u}\}$ and $\{D_{(t-u,\ t]}\}$, where $D_{(t-u,\ t]}=N_t-N_{t-u}$ for a lead time $u>0$, are mutually independent of each other, the analysis of net inventory process $\{NIS_t;\ t\geq 0\}$ will become straightforward, from which the cost process can be immediately derived whose average one may seek. The net inventory process is defined as $NIS_t=IP_{t-u}-D_{(t-u,\ t]}$.

The primary objective of this study is to prove that $\{IP_{t-u};\ t-u\geq 0\}$ and $\{D_{(t-u,\ t]}\}$ are mutually independent of each other, even in the case of non-standard inventory models with non-stationary Poisson demand processes.

Among references, only the inventory position process associated with stationary Poisson demand process has been specified in Hadley and Whitin [1].

2. Proof of Mutual Independence

From the point of view of the mathematical theory of probability a stochastic process is

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best defined as a famility $\{X(t); t \in T\}$ of random variables, where the parameter set T is called the index set of the process.

When demands arrive at time points $t_1, t_2, \cdots\cdots, (0 < t_1 < t_2)$, the successive inter-arrival times $\{X_i; i > 1\}$ are defined as $X_1 = t_1$, $X_2 = t_2 - t_1$, $\cdots\cdots$, $X_n = t_n - t_{n-1}$, $\cdots\cdots$. Let N_t be cumulative demand by time t, $t \geqq 0$. Then $\{N_t;\ t \geq 0\}$ is a discrete-valued continuous-parameter stochastic process with sample paths increasing in unit steps.

An inventory position IP_t at time t totally depends upon the demand process $\{D_t;\ t \in T\}$. If an inventory system is started with $IP_o = r + i (i = 1, 2, \cdots\cdots, Q)$ at time $t = 0$, then $IP_{t-\tau} = r + j\ (j = 1, 2, \cdots\cdots, Q)$ at time $t - \tau > 0$ can be reached after the $(i-j)^+$ or $\{i + (m-1)Q + (Q-j);\ m = 1, 2, \cdots\cdots\}$ demand materialization by time $t - \tau$, where m denotes the total number of order placements by time $t - \tau$ and

$$(i-j)^+ = \max\{o, i-j\}.$$

Suppose now that we consider the sequnce of events consisting of the times at which an order in the amount of Q is placed and received in the constant lead time τ. Defining Y_k to be the time elapsed between the $(k-1)^{th}$ and k^{th} orders, the sequence of random variables $\{Y_k;\ k = 1, 2, \cdots\cdots\}$ forms a modified renewal process in which the distribution functions are given by

$$P\{Y_1 \leq y_1\} = P\{S_i \leq y_1\} \equiv F_i(y_1) = P\{N_{y_1} \geq i\},$$

where $\begin{cases} i = \text{the initial stock over the reorder point } r. \\ S_i = \text{the renewal epoch of the } i^{th} \text{ demand and so equal to } \sum_{k=1}^{i} X_k \\ F_i(\cdot) = \text{the n-fold convolution of the identical distribution } F \text{ of } \{X_i\}, \end{cases}$

and likewise,

$$P\{Y_k \leq y_k\} = P\{S_Q \leq y_k\} = P\{N_{y_k} \geq Q\} = F_Q(y_k), \text{ for } k = 2, 3, \cdots\cdots,$$

$$\text{since } \{Y_k \leq y_k\} \Longleftrightarrow \{(S_{i+(k-1)Q} - S_{i+(k-2)Q}) \leq y_k\}$$

$$\Longleftrightarrow \{S_Q \leq y_k\} \text{ for } k = 2, 3, \cdots\cdots.$$

Thus, a new renewal process $\{W_m;\ m = 0, 1, 2 \cdots\cdots\}$ is defined such that

$$W_o = Y_o = 0$$

$$W_m = \sum_{k=1}^{m} Y_k = S_{i+(m-1)Q}, \quad m = 1, 2, 3, \cdots,$$

where "m=0" means that no order is placed yet.

Let $(t - \tau - \theta)$ and m be, respectively, particular values of the time T and the serial number M of the last order placed no later than $t - \tau$. If we assume that $IP_{t-\tau} = r + j (j = 1, 2, \cdots\cdots, Q)$ at time $t - \tau$, then we see that $(Q - j)$ demands are further needed in the time interval $(t - \tau - \theta, t - \tau]$, for $\theta \geq 0$, since the inventory position at time $t - \tau - \theta$ is $r + Q$ immediately after the m^{th} order is placed at time $t - \tau - \theta$.

Let $Z_{t-\tau}$ be the time from $t - \tau$ until the first demand subsequent to $t - \tau$, that is,

$$Z_{t-\tau} = S_{N_{t-\tau}+1} - (t - \tau),$$

where

$$S_{N_{t-\tau}} \leq t - \tau < S_{N_{t-\tau}+1}.$$

The variable $Z_{t-\tau}$ will be the residual or excess waiting time at epoch $t - \tau$. Then, the distribution function of $Z_{t-\tau}$ can be determined by use of the renewal equation for $m(t) = E\{N_t\}$.

Furthermore, let $t-\tau+Z$ be the time point at which the first demand occurs after time $t-\tau$. Then, the random variable $Z_{t-\tau}$ may have a different distribution from those of X_i's. However, the distribution of $D_{(t-\tau,\, t]}$ is determined by partitioning in accordance with the time $t-\tau+Z$ at which the first demand occurs after the time $t-\tau$ and the time interval $(t-\tau+Z,\, t]$ in which $k-1$ demands occur. For the explicit form of the distribution of $\{Z_{t-\tau}\}$ and $\{D_{(t-\tau,\, t]}\}$, refer to Sung [2].

From the preceding discussions we can see that $P\{IP_{t-\tau}=x\}$ is a function of $P\{N_{t-\tau}=y\}$, which means the inventory position $IP_{t-\tau}$ is determined by $N_{t-\tau}$. Thereby, we shall prove that given $IP_0=r+i\,(i=1,\, 2,\, \cdots\cdots,\, Q)$ at time $t=0$, for non-stationary Poisson demand process, the distribution of $IP_{t-\tau}$ is independent of that of $D_{(t-\tau,\, t]}$, even though $D_{(t-\tau,t]}=N_t-N_{t-\tau}$.

Theorem 2.1.

For the continuous-review $\langle Q\,,r\rangle$ inventory system with backorders allowed, constant lead time $\tau\geq 0$, demands occurring in accord with a non-stationary Poisson process with finite mean, and with $IP_0=r+i\,(i=1,\, 2,\, \cdots\cdots,\, Q)$,

$$P\{IP_{t-\tau}=r+j,\ D_{(t-\tau,\, t]}=k\}=P\{IP_{t-\tau}=r+j\}P\{D_{(t-\tau,\, t]}=k\},$$
$$\text{for } j=1,\, 2,\, \cdots\cdots,\, Q \text{ and } k=0,\, 1,\, 2,\, \cdots\cdots.$$

Proof:

$$P\{IP_{t-\tau}=r+j,\ D_{(t-\tau,\, t]}=k\}$$
$$=\sum_{m=0}^{\infty}\int_{\theta=0}^{\theta=t-\tau}P\{IP_{t-\tau}=r+j,\ D_{(t-\tau,\, t]}=k\,|\,M=m,\ T=t-\tau-\theta\}\,d\phi_m\{T<t-\tau-\theta\}$$
$$=\begin{bmatrix}P\{N_{n-\tau}=(i-j),\ N_t-N_{t-\tau}=k\}^{+}+\\ \sum_{m=1}^{\infty}\int_{\theta=0}^{\theta=t-\tau}P\{N_{t-\tau}-N_{t-\tau-\theta}=Q-j,\ N_t-N_{t-\tau}=k\,|\,M=m,\ T=t-\tau-\theta\}\cdot dP\{W_m<t-\tau-\theta\}\end{bmatrix}$$
$$\text{where,}\qquad P\{N_{t-\tau}=(i-j),\ N_t-N_{t-\tau}=k\}^{+}=0\ \text{ if } i<j.$$
$$=\begin{bmatrix}\mathrm{P}\{N_{t-\tau}=(i-j),\ N_t-N_{t-\tau}=k\}^{+}+\\ \sum_{m=1}^{\infty}\int_{\theta=0}^{\theta=t-\tau}P\{N_{t-\tau}-N_{t-\tau-\theta}=Q-j,\ \ N_t-N_{t-\tau}=k\}\,dP\{W_m<t-\tau-\theta\}\end{bmatrix}$$
$$=\left\{\begin{array}{l}\int_{Z=0}^{Z=\tau}P\{N_{t-\tau}=(i-j),\ N_t-N_{t-\tau}=k\,|\,Z_{t-\tau}=Z\}^{+}\cdot dP\{Z_{t-\tau}<Z\}+\\ \sum_{m=1}^{\infty}\int_{\theta=1}^{\theta=t-\tau}\int_{Z=0}^{Z=\tau}P\{N_{t-\tau}-N_{t-\tau-\theta}=Q-j,\\ \qquad\qquad N_t-N_{t-\tau}=k\,|\,Z_{t-\tau}\leq Z\}\cdot dP\{Z_{t-\tau}\leq Z\}\cdot dP\{W_m\leq t-\tau-\theta\}\end{array}\right\}$$
$$=\begin{bmatrix}\int_{Z=0}^{Z=\tau}P\{N_{t-\tau}=(i-j),\ N_{t-\tau}=k-1\}^{+}\cdot dP\{Z_{t-\tau}\leq Z\}+\\ \sum_{m=1}^{\infty}\int_{\theta=0}^{\theta=t-\tau}\int_{Z=0}^{Z=\tau}P\{N_\theta=Q-j,\ N_{\tau-Z}=k-1\}\cdot dP\{Z_{t-\tau}<Z\}\cdot dP\{W_m\leq t-\tau-\theta\}\end{bmatrix}$$
$$=\left\{\begin{array}{l}\int_{Z=0}^{Z=\tau}P\{N_{t-\tau}=(i-j)\}^{+}\cdot P\{N_{\tau-Z}=k-1\}\cdot dP\{Z_{t-\tau}\leq Z\}+\\ \sum_{m=1}^{\infty}\int_{\theta=0}^{\theta=t-\tau}\int_{Z=0}^{Z=\tau}[P\{N_\theta=Q-j\}\cdot P\{N_{\tau-Z}=k-1\}]\cdot dP\{Z_{t-\tau}\leq Z\}\\ \qquad\qquad\cdot dP\{W_m\leq t-\tau-\theta\}\end{array}\right\}$$
$$=\begin{bmatrix}P\{N_{t-\tau}=(i-j)\}^{+}\cdot\int_{Z=0}^{Z=\tau}P\{N_{\tau-Z}=k-1\}\cdot dP\{Z_{t-\tau}<Z\}+\\ \sum_{m=1}^{\infty}\int_{\theta=0}^{\theta=t-\tau}P\{N_\theta=Q-j\}dP\{W_m\leq t-\tau-\theta\}\int_{Z=0}^{Z=\tau}P\{N_{\tau-Z}=k-1\}\cdot dP\{Z_{t-\tau}<Z\}\end{bmatrix}$$

$$= \left[\begin{array}{l} P\{N_{t-\tau}=(i-j)\}^{+}+\sum_{m=1}^{\infty}\int_{\theta=0}^{\theta=t-\tau} P\{N_{\theta}=Q-j\}\cdot dP\{W_m<t-\tau-\theta\} \\ \qquad \cdot \int_{Z=0}^{Z=\tau} P\{N_{\tau-Z}=k-1\}\cdot dP\{Z_{t-\tau}\leq Z\} \end{array} \right]$$

$$=P\{IP_{t-\tau}=r+j\}\cdot P\{D_{(t-\tau,t]}=k\}$$

from the result of the distributions of $\{IP_{t-\tau}\}$ and $\{D_{(t-\tau,t]}\}$ derived in Sung [2].

∴ The proof is complete.

The above analysis can be directly cited for the periodic-review inventory models, $\langle nQ, r, T\rangle$ and $\langle R, r, T\rangle$. Recall that under both models an order is placed at a review time T_k ($k=0, 1, 2, \cdots\cdots,$) if and only if the inventory position IP_{T_k} of the system is less than or equal to r. Therefore, the exactly same procedure treated in the proof of Theorem 2.1 will give rise to the next theorem.

Theorem 2.2

For the periodic-review inventory systems of $\langle nQ, r, T\rangle$ and $\langle R, r, T\rangle$ with the same restrictions placed in Theorem 2.1 and $\xi\geq 0$,

$$P\{IP_{T_k}=r+j,\ D_{(T_k,\ T_k+\xi]}\}=P\{IP_{T_k}=r+j\}P\{D_{(T_k,\ T_k+\xi]}=m\},$$

for $m, k=0, 1, 2, \cdots\cdots,$ and $j=1, 2, \cdots\cdots, Q(R-r$ for $\langle R, r, T\rangle)$.

3. Deriving Limit Distribution of Net Inventory Position Processes

In view of the system analysis discussed in Introduction, it is defined that for $\langle Q, r\rangle$ systems,

$$NIS_t=IP_{t-\tau}-D_{(t-\tau,t]} \text{ for } t\geq\tau\geq 0$$
$$=OH_t-BO_t$$

and so

$$NIS_t=OH_t, \text{ if } NIS_t\geq 0$$
$$=BO_t, \text{ otherwise.}$$

Furthermore, from the result of Theorem 2.1,

$$P\{NIS_t=r+s\}=\sum_{j=1}^{Q}P\{IP_{t-\tau}=r+j,\ D_{(t-\tau,t]}=j-s\}^{+},$$

for $s=Q, Q-1, \cdots\cdots, 0, -1, 2, \cdots\cdots,$

$$=\sum_{j=1}^{Q}P\{IP_{t-\tau}=r+j\}P\{D_{(t-\tau,\ t]}=j-s\}^{+}$$

where

$$P\{D_{(t-\tau,\ t]}=j-s\}^{+}=P\{D_{(t-\tau,\ t]}=j-s\}, \text{ if } j\geq s$$
$$=0 \qquad \text{, otherwise.}$$

Now, consider the distribution of $IP_{t-\tau}$ processes. Under the assuming conditions discussed in Theorem 2.1, it follows that

$$P\{N_t=n\}=P\{S_n\leq t-\tau\}-P\{S_{n+1}\leq t-\tau\},\ n=1, 2, \cdots\cdots,$$
$$=F_{S_n}(t-\tau)-F_{S_{n+1}}(t-\tau)$$
$$=\int_0^{t-\tau}e^{-m(s-\tau)}\frac{\{m(s-\tau)\}^{n-1}}{(n-1)!}\cdot\frac{\{n-m(s-\tau)\}}{n}\gamma(s-\tau)ds,$$

where

$$m(t)=E\{N_t\}$$
$$\gamma(t)=\frac{d}{dt}m(t)$$
$$f_{S_n}(t-\tau)e^{-m(t-\tau)}\frac{\{m(t-\tau)\}^{n-1}}{(n-1)!}\gamma(t-\tau).$$

This relation indicates that as long as the so-called mean value function $m(t)$ associated with N_t processes is identified, the corresponding limit distribution of $IP_{t-\tau}$(or IP_{T_k}) can be easily determined.

For $\langle nQ, r, T\rangle$ and $\langle R, r, T\rangle$ systems, such derivations can be made regarding the following relations;

$$NIS_{T_{k+\xi}}=IP_{T_k}-D(T_k, T_{k+\xi}]=OH_{T_{k+\xi}}-BO_{T_{k+\xi}}.$$

4. Conclusion

It is clear that the discussions in section 2 can directly lead to some analytical cost model constructions of related stochastic inventory systems $\langle Q, r\rangle$, $\langle nQ, r, T\rangle$ and $\langle R, r, T\rangle$ with demands occurring in accord with a non-stationary Poisson process, having finite mean, in terms of the probability distributions OH_t and BO_t.

Therefore, for the implementation of this work the only thing to do is stochastically to characterize the mean value functions of demand processes under real non-stationary circumstances.

Finally, recall that this study was restricted to the inventory systems having non-stationary Poisson demand processes. However, the subject associated with more general demand processes is still open to question.

REFERENCES

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