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—Research Paper—

A Multi-period Behavioral Model for Portfolio Selection Problem

G. Pederzoli and R. Srinivasan*

Abstract

This paper is concerned with developing a Multi-period Behavioral Model for the portfolio selection problem. The unique feature of the model is that it treats a number of factors and decision variables considered germane in decision making on an interrelated basis. The formulated problem has the structure of a Chance Constrained programming Model. Then empoloying arguments of Central Limit Theorem and normality assumption the stochastic model is reduced to that of a Non-Linear Programming Model. Finally, a number of interesting properties for the reduced model are established.

SECTION 1 : INTRODUCTION

Modelling the classical portfolio selection problem has received the repeated attention of many researchers for well over a decade. Consequently, four different approaches have emerged and are discernible in the literature. The first of these is the mean varince approach pioneered by Markowitz (1) The second is a Chance Constrained programming Approach initiated by Nausland and Whinston (2) The third approach has its origin in capital growth consideration and is due to Latane (3) and Breiman (4) The fourth approach is based upon Statistical Decision Theory and is due to Mao and Sarndal (5) All these approaches will be reviewed in this paper briefly.

Markowitz (1) developed a quadratic programming model to determine the set of efficient portfolios which provides the maximum return for every possible level of risk and minimizes the risk for every level of return. It is left to the individual investor to select a single efficient set which best serves his needs. By extending the theory of Markowitz further one can show that under conditions of risk free interest the investor need only lever himself up or down by borrowing or lending to obtain a position consistent with his attitude towards risk vis-a-vis expected return.

Nausland and Whinston (2) assumed that the investor has a known income stream up to a certain time horizon. Further, in each period he will invest the difference between his in-

*Concordia University, Montreal, Carada.

come and consumption in the stock market. His goal is to maximize the expected return at the end of each time horizon subject to the following constraints:

(a) a risk constraint which states that losses must be stochastically less than a specified amount and

(b) a capital constraint which probabilistically stipulates that the invested capital should be below a given limit. This limit will vary according to accumlated capital gains.

The resultant model has the structure of a chance constrained programming model with a number of input parameters to be estimate.

Capital growth approach is a sequential portfolio approach which is developed using the assumptions of Markowitz model with the exception that a rational investor would prefer more money to less money at all time (6). This approach includes the possibility of investing in other less risky opportunities such as bonds etc.

Mao and Sarndal (5) restructured the Markowitz model within the framework of a statistical decision theory. In this, the future returns from securities are considered as a function of unknown states of nature. The investor has certain prior probabilities for these states which he modifies in the light of new experimental information using the Bayesian technique. Finally, he selects a porfolio of securities which maximizes the weighted average pay off using as weights the posterior probabilities of the states of nature.

In all these approaches, variance is used as the only measure of risk. This has been open to question from the fact that variance assumes that deviations both above and below the return are equally undesirable. This however, is not true. Further these approaches disregard completely the magnitude of potential loss which occurs due to the timing of investment. This loss is of considerable importance to an investor since sudden changes in the market prices of stocks due to the market turning bear could jeopardize his capacity to survive in the long run.

Again the effects of states of nature is completely ignored in these approaches, except in the decision theory formulation. But even there the treatment is at a very elementary level. The return from a security is influenced by the environmental factors such as future business conditions, the course of fiscal and monetary policies and international political and econmic conditions. Not all these factors will simultaneously influence the performance of securities but whatever be the critical factors in a given situation, a model of portfolio selection will be conceptually more meaningful if it forces the investor to explicitly state his views regarding these factors. Cohen and Elton (7) point out that the effects of the market can be ignored only when a state of the world does not differentially affect the relative amout of risks of various securities and the market is such that the states of the world do not change the risk return preference of an investor.

Finally, cost of transactions, which is incurred when modifying or revising the present portfolio, is totally ignored in all these approaches. Pogue (8) reports that on an average it costs 2 to 4 percent of the investors' capital to modify the portfolio. In fact, any decision to modify the portfolio should be conditioned by the transaction costs involved.

It is therefore the primary concern of this paper:

(a) to identify all the factors which are regarded as relevant to the portfolio management problem and need to be considered for effective decision making, and

(b) using these factors and the various objectives an investor desires to achieve to develop a model of effective portfolio selection.[1]

The remainder of this paper is presented in four sections. In section Ⅱ, we identify a number of decision factors which are considered germane for effective decision making and justify their inclusion in the model building. In section Ⅲ we develop a mulit-period behavioral model for allocating the resources among the selected stocks. This will be a Chance Constrained Programming Model and will include the effects of the following:

(1) States of nature.
(2) Cost of transaction.
(3) Uncertainty in the Prediction process.
(4) Growth consideration, and
(5) Opportunity lost.

In section IV, using probabilistic arguments the stochastic model is reduced to a deterministic non-linear programming model. Finally, in section V we establish the convexity property for the reduced model. The model is in a known form for which a number of algorithms exist in the literature of Operations Research.

SECTION Ⅱ : FACTORS INVOLVED IN THE DECISON MAKING

All stocks when traded in the stock market are mostly done through stock brokers and this results in the payment of transaction costs. It consists of the following two parts:

1. Broker's Commission: It is the broker's fee for purchasing or selling securities for a client. It primarily depends upon the price of shares traded in the market. At the end of 1969, the average price of a share of stock listed on the stock exchange (N.Y.) was approximately $50. So, the average purchase and sale transaction would therefore cost the investor's capital to be reduced by about 2% of the assets involved.[2]

2. Additional Commission for Exchange Acquisition and Distribution: Exchange acquisition is a method of filling an order to buy a large block of stocks while exchange distribution involves disposing it on the floor of a stock exchange.

In the case of Exchange Acquisition the price of a stock to the buyer may be on a net basis or on a commission basis. For exchange distribution a special commission is usually

1) The phrase "effective portfolio" refers to the selection of portfolio which will satisfy the specific requirements of an investor.

2) The brokerage fee on a 100 share transaction for securities are summarized in the following table:

Share Price	Value of Stock	Total Trading Cost	Percentage Value of Stock
$200	$20,000	$123.40	0.62%
100	10,000	103.20	1.03
50	5,000	93.10	1.86
10	3,000	37.39	3.38
3	300	20.01	6.67

New York Stock Exchange Brokerage Fee Schedule Source: Cohen and Zinbarg, Investment Analysis and Portfolio Management.

paid by the seller and the buyer normally pays no commission. The above source of transaction cost are of primary interest to a large institutional investor. The costs take the form of either unfavorable price discounts or the premiums the investor may have to pay. They normally result from the cost of informing additional purchasers or sellers about the unusual opportunities that exist and offering them inducements to rebalance their portfolios which consists of favorable price spreads etc. Thus any model which is relevant for portfolio selection should include not only the brokerage component of transaction cost but also any incremental cost the investor expects to be associated with large transactions. It should be clear from the above exposition that whenever a decision with regard to portfolio selection or revision is made, an investor should make sure that the expected incremental benefit is at least as high as to compensate for the cost of transaction. Unforunately, most of the existing models assume that it is costless to modify a portfolio that is effective in terms of the revised expectations about security performance. This may only result in poor decision making.

Investor's Objective

The often repeated objective of maximizing the return from a portfolio for every possible levels of risk has severe limitations in terms of efficiency and conceptual completeness. For example Markowitz model is a point in time analysis (9). It is run at a single time period. The selected portfolio is bought and remains unchanged until the next run. This introduces problem in the choice of a time period for consideration. The longer the time period between runs the farther the portfolio may drift from the efficient region. Also, this may result in sharp decline in the prices of stocks included in the portfolio due to the market turning bear. But if the chosen time period is short the problem of data collection may become impracticable due to the paucity of time and limitations in computer storage transaction cost which may offset the benefits of expected higher prices of stocks.

So, the decision with regard to selection of a portfolio should be made only after choosing an appropriate length of the planning period and with the requirement that the decision is irrevocable until the end of the period (10).

Thus the existing objectives are inadequate. A more realistic objective would be one which not only maximizes the total return from a porfolio but laso minimizes the opportunity loss (loss in the potential return). In this paper an objective function will be developed which will meet these requirements.

Opportunity Loss

An important consequence to the choice of time period is the incurrence of opportunity loss. We shall provide a formal definition here.

Definition: Opportunity loss, a random variable, is defined as the loss incurred due to the choice of a review period. This is an unavoidable loss since the "lows" ("highs") of all the stocks under consideration will not coincide exactly with the beginning (ending) of the period.

Unfortunately, the static models do not consider this cost as important. This can be only at the cost of accuracy.

States of Nature

The effects of the State of Nature on the decision making is much too important to be ignored in the model building for portfolio selection. There are enough evidence to show that the market conditions do affect the return from a stock. Cohen and Elton (7) point out that the effects of state of nature can be ignored only when it does not change the risk return preferences of an investor and secondly, when it does not affect the relative amounts of risks of various securities. Mao and Sarndal (5) also point out that the future returns from securities should be viewed as the function of unknown states of nature.

So it is imperative that the effects of the states of nature should be quantified and incorporated in model building. Surprisingly enough, very few authors have given due consideration to this.

Growth Consideration

Any model for portfolio selection should include the growth factor. Growth can occur due to two reasons.

(1) Money may be added and withdrawn during the beginning of the period, and

(2) Previous investments may be carried over with accumulated dividends and price appreciations.

Uncertainty and Levels of Confidence

The fluctuations in the stock market are due to a numer of factors some of which are identifiable and some are not. So any decision process must take cognizance of this. Predicting the future stock trends with certainty is out of question. Surprisingly enough, most of the models either assume perfect certainty or at best some arbitrary levels of confidence. While the former is unrealistic the latter can be damaging, since over estimating one's ability can have undesirable consequences in terms of cost and efficiency. Hence a model for portfolio selection should include this factor.

SECTION Ⅲ : THE MODEL

In this section we shall develop a Multi-period Behavioral model for effective portfolio selection problem, The model will treat simultaneously the effects of the various factors discussed in Section Ⅱ, It will be in the form of a chance constrained programming model wherein the effects of the factors will appear as input parameters. Different methods are available in the literature for evaluating these parameters (11). Before developing the model we need to define:

(a) the objective of an investor, and

(b) the assumptions required for the development of the model. They are as follows.

Objective

A rational investor, in this paper, is defined to be one who wants to maximize the expected total return from an investment over the effects of the various possible states of nature after adjusting it for the cost of transaction and expected opportunity loss.

Assumptions

(1) The stock market can be characterized as consisting of various states of nature. The market stays for a random length of time in each state before making a transition to the next state.

(2) Return from a stock is a function of the state of nature and includes both dividend and capital appreciation of stock.

(3) Money can be borrowed and lent at a risk free interest rate.

(4) Existence of one or more risky opportunities and perfect liquidity and divisibilty of assets at each decision point.

Except for assumption (1) the rest of them are common for all the existing models in the literature, whether deterministic or stochastic and single period or multiperiod and hence need not be discussed here (6).

For the first assumption in addition to the brief explanation in section II the reader is referred to the recent work of the authors about the implications of stock market trends on the prices of stocks (12).

Notation

$R_{ij,L}$=return per unit of investment (Random Variable) from the i^{th} stock under L^{th} state of nature.

$i=1\cdots\cdots n$ (after being adjusted for the $L=1\cdots\cdots m$ cost of transaction).

$\pi_{jL}=\pi_L$=ergodic probability for the L^{th} state of nature.

$L=1\cdots\cdots M$

X_{ij}=proportion of the amount of money to be invested in the i^{th} stock during the j^{th} period.

$i=1\cdots\cdots n \qquad j=1\cdots\cdots N$

$O_{ij,L}$=opportunity loss per unit of investment (Random Variable) from the i^{th} stock under L^{th} state of nature during the j^{th} period.

$i=1\cdots\cdots n \qquad j=1\cdots\cdots N \qquad L=1\cdots\cdots m$

R_{ij-1}=return from the i^{th} stock during the $j-1^{th}$ period. It is a known amount.

$i=1\cdots\cdots n \qquad j=1\cdots\cdots N$

α_j=level of confidence for the j^{th} period for the return constraint.

$j=1\cdots\cdots N$

β_j=level of confidence for the j^{th} period for the opportunity loss constraint.

$j=1\cdots\cdots N$

A_j=minimum return an investor expects from his investment during the j^{th} period.

$j=1\cdots\cdots N$

B_j=maximum opportunity loss an investor can afford during the j^{th} period.

$j=1\cdots\cdots N$

N_j=amount of money added to the available capital for investment at the beginning of the j^{th} period.

$j=1\cdots\cdots N$

S_j = amount of money withdrawn from the available capital for investment at the beginning of the j^{th} period.

$j=1\cdots\cdots N$

Model

$$\text{Max } E\{\Sigma^n{}_{i=1}\Sigma^M{}_{L=1}\pi_L R_{ij},{}_L X_{ij}-\Sigma^n{}_{i=1}\Sigma^M{}_{L=1L}O_{ij},{}_L X_{ij}\}$$

Subject to:

Constraint 1.

$$\Pr\{\Sigma^n{}_{i=1}\Sigma^M{}_{L=1}\pi_L R_{ij},{}_L X_{ij}\geq A_j\}\geq\alpha_j \qquad j=1\cdots\cdots N$$

Constraint 2.

$$\Pr\{\Sigma^n{}_{i=1}\Sigma^M{}_{L=1}\pi_L O_{ij},{}_L X_{ij}\geq B_j\}\geq\beta_j \qquad j=1\cdots\cdots N$$

Constraint 3.

$$\Sigma^n{}_{i=1}X_{ij}\leq N_j-S_j+\Sigma^n{}_{i=1}X_{i,j-1}+\Sigma^n{}_{i=1}R_{ij-1}X_{ij-1} \qquad j=1\cdots\cdots N$$

Constraint 4.

$$X_{ij}\geq 0 \quad \forall_{i,j} \qquad i=1\cdots\cdots n \quad j=1\cdots\cdots N$$

Explanation

Constraint 1. states that the total earning from investment during the j^{th} period, should exceed probabilistically a certain given amount (in dollars).

Constraint 2. is probabilistic and states that the total opportunity loss during the j^{th} period should be below a given amount (in dollars).

Constraint 3. is deterministic which imposes a ceiling on the amount invested (in dollars) in the stock market and includes addition and withdrawals.

Constraint 4. is a deterministic non-negativity requirement.

The objective is to maximize the expected return (adjusted for the cost of transaction) summed over all stocks and states of nature minus the expected opportunity loss summed over all stocks and states of nature.

SECTION Ⅳ : MODEL REDUCTION

Reducing the chance constrained programming model to a known deterministic form requires the following steps to be accomplished:

Step 1: Substitute the values of the computed parameters in the model.

Step 2: Make the necessary assumptions and substitutions for reducing the stochastic model to a deterministic model and justify the assumptions using probabilistic arguments.

Step 3: Reduce the stochastic model to a deterministic nonlinear programming model.

These steps will be achieved sequentially.

Step 1

Substitute the values of the parameters in the model of section Ⅱ.

Notation

π_L = be the computed probability for the L^{th} state of nature.

$L=1, 2\cdots\cdots M$

A_j = be the computed level of confidence for the return constraint, for the j^{th} R.R. period.

$j=1, 2\cdots\cdots N$

β_j = be the computed level of confidence for the opportunity loss constraint, for the j^{th} R.R. period.

$j=1, 2\cdots\cdots N$

The other notations are the same as in section Ⅱ and so will not be repeated here. We shall denote the resulting problem as 4. 1.

Problem 4. 1

Objective:

$$\text{Maximize } E\{\Sigma^M{}_{L=1}\Sigma^n{}_{i=1}R_{ij},{}_L\pi_L X_{ij}-\Sigma^M{}_{L=1}\Sigma^n{}_{i=1}O_{ij},{}_L\pi_L X_{ij}\}$$

Subject to:

1. $\Pr\{\Sigma^M{}_{L=1}\Sigma^n{}_{i=1}R_{ij},{}_L\pi_L X_{ij}\geq A_j\}\geq\alpha_j$ for a fixed j
2. $\Pr\{\Sigma^n{}_{i=1}\Sigma^M{}_{L=1}O_{ij},{}_L\pi_L X_{ij}\leq B_j\}\geq\beta_j$ for a fixed j
3. $\Sigma_i X_{ij}\leq N_j-S_j+\Sigma^n{}_{i=1}R_{ij-1}X_{i,j-1}+\Sigma^n{}_{i=1}X_{i,j-1}$ for a fixed j

And

4. $X_{ij}\geq 0$ for all $i=1, 2\cdots\cdots n$ and a fixed j.

Step 2

Assumptions and Substitutions

Let,

$$\Sigma_L R_{ij},{}_L\pi_L=C_{ij} \text{ for a fixed } j$$

$$\Sigma_L O_{ij},{}_L\pi_L=D_{ij} \text{ for a fixed } j$$

Problem 4. 1 can be written as:

$$\text{Maximize } E\{\Sigma^n{}_{i=1}C_{ij}X_{ij}-\Sigma^n{}_{i=1}D_{ij}X_{ij}\}$$

Subject to:

$$\Pr\{\Sigma^n{}_{i=1}C_{ij}X_{ij}\geq A_j\}\geq\alpha_j \text{ for a fixed } j$$

$$\Pr\{\Sigma^n{}_{i=1}D_{ij}X_{ij}\leq B_j\}\geq\beta_j \text{ for a fixed } j$$

$$\Sigma^n{}_{i=1}X_{ij}\leq N_j-S_j+\Sigma_i C_{ij-1}X_{ij-1}+\Sigma_i X_{i,j-1} \text{ for a fixed } j$$

$$X_{ij}\geq \text{for all } i \text{ and a fixed } j$$

Let us make the following substitutions and assumptions in the model.

Substitution

Let, (1) $E(C_{1j})=\mu_{1j}$ $E(C_{2j})=\mu_{2j}\cdots\cdots E(C_{nj})=\mu_{nj}$

Where μ_{ij} is the average expected return from the i^{th} stock during the j^{th} period.
$i=1\cdots\cdots n \quad j=1\cdots\cdots N$

(2) Variance $(C_{1j})=\sigma^2{}_{1j}$ Variance $(C_{2j})=\sigma^2{}_{2j}\cdots\cdots$Variance $(C_{nj})=\sigma^2{}_{nj}$
Where $\sigma^2{}_{ij}$ is the variance of the i^{th} stock during the j^{th} period.
$i=1\cdots\cdots n \quad j=1\cdots\cdots N$

(3) $E(D_{1j})=\gamma_{1j}$ $E(D_{2j})=\gamma_{2j}\cdots\cdots E(D_{nj})=\gamma_{nj}$
Where γ_{ij} is the average expected loss from the i^{th} stock during the j^{th} period.
$i=1\cdots\cdots n \quad j=1\cdots\cdots N$

(4) Variance $(D_{1j})=W^2{}_{1j}$ Variance $(D_{2j})=W^2{}_{2j}\cdots\cdots$Variance $(D_{nj})=W^2{}_{nj}$
Where $W^2{}_{ij}$ is the variance in the opportunity loss associated with the j^{th} stock during the j^{th} period.
$i=1\cdots\cdots n \quad j=1\cdots\cdots N$

Assumptions

(1) The functional form of the probability distribution of $(\Sigma_i C_{ij}X_{ij})$ and $(\Sigma_i D_{ij}X_{ij})$ for any fixed j is known and is completely determined by the fractiles of the distribution.

Theorem 4.1

If C_{1j}, $C_{2j}\cdots\cdots C_{nj}$ are independently normally distributed random variables with means μ_{1j}, $\mu_{2j}\cdots\cdots\mu_{nj}$ and Variances $\sigma^2{}_{1j}$, $\sigma^2{}_{2j}\cdots\cdots\sigma^2{}_{nj}$ respectively then,

$$Y=\Sigma^n{}_{i=1}C_{ij}X_{ij}$$

Where X_{ij}'s are constants, has the normal distribution with mean $\Sigma_i\mu_{ij}X_{ij}=M_j$ and Variance $\Sigma_i\sigma^2{}_{ij}X^2{}_{ij}=\Sigma_j$

Proof

Given: $E(C_{ij})=\mu_{ij}$ for all i and any j
Variance: $(C_{ij})=\sigma^2{}_{ij}$ for all i and any j
Covariance: $(C_{ij}C_{kj})=0$ for all $i\neq k$ and any j

Then by definition, $E(\Sigma_i X_{ij}C_{ij})=\Sigma_i X_{ij}E(C_{ij})=\Sigma^n{}_{i=1}X_{ij}\mu_{ij}$ (4.1)

Variance $(\Sigma_i X_{ij}C_{ij})^2=\Sigma_i X^2{}_{ij}C^2{}_{ij}$ (4.2)

The characteristic function of $C_{kj}X_{kj}$ is given by (13).

$$\Phi_{C_{kj}X_{kj}}(t)=\mathrm{Exp}\left(it\ X_{kj}\mu_{kj}-\frac{t^2\sigma^2{}_{kj}}{2}X^2{}_{kj}\right) \text{ for any } k \text{ and fixed } j \quad (4.3)$$

Then,

$$\Phi_Y(t)=\Phi^{(t)}_{\Sigma^n_{k=1}C_{kj}X_{kj}}=\mathrm{Exp}\left(it\ X_{1j}\mu_{1j}-\frac{t^2\sigma^2{}_{1j}X_{1j}}{2}\right)$$

$$\mathrm{Exp}\left(it\ X_{2j}\mu_{2j}-\frac{t^2\sigma^2{}_{1j}X^2{}_{2j}}{2}\right)\cdots\cdots$$

$$\mathrm{Exp}\left(it\ X_{nj}\mu_{nj}-\frac{t^2\sigma^2{}_{nj}X^2{}_{nj}}{2}\right) \quad (4.4)$$

Rearranging the terms in (4.4) we get,

$$=\mathrm{Exp}[it\ (X_{1j}\mu_{1j}+X_2\mu_{2j}+\cdots\cdots+X_{nj}\mu_{nj})]-\frac{t^2}{2}[(\sigma^2{}_{1j}X^2{}_{1j}+\sigma^2{}_{2j}X^2{}_{2j}+\cdots\cdots+\sigma^2{}_{nj}X^2{}_{nj})]$$

Thus $\Sigma_i C_{ij}X_{ij}$ is distributed normally with mean $\Sigma_i\mu_{ij}X_{ij}$ and variance $\Sigma_i\sigma^2{}_{ij}X^2{}_{ij}$.

Theorem 4.2

If D_{1j}, $D_{2j}\cdots\cdots D_{nj}$ are independently normally distributed random variables with means γ_{1j}, $\gamma_{2j}\cdots\cdots\gamma_{nj}$ and variances W^2_{1j}, $W^2_{2j}\cdots\cdots W^2_{nj}$ respectively then:

$$Z=\Sigma_i D_{ij}X_{ij}$$

Where X_{ij}'s are constants, has the Normal distribution with mean $\Sigma_i\gamma_{ij}X_{ij}=\Gamma_j$ and variance $\Sigma_i W^2_{ij}X^2_{ij}=\Omega_j$

Proof

The steps of the proof are the same as in theorem 4.1 and hence will not be repeated here. If the theorem 4.1 and 4.2 do not hold for this problem we can still take recourse to normal distribution by way of central limit theorem approximation. The corresponding theorem is as follows:

Theorem 4.3

If the individual random variables C_{1j}, C_{2j}, $\cdots\cdots$, C_{nj} have an arbitrary distribution then $(\Sigma_i C_{ij}C_{ij})$ is distributed approximately normal for any fixed j.

Proof

Refer to Hillier [14].

Assumption 2

Whatever may be the distribution of $\Sigma_i C_{ij}X_{ij}$ and let F(.) denote the cumulative distribution function of:

$$\frac{(\Sigma_i C_{ij}X_{ij}-A_j)-E(\Sigma_i C_{ij}X_{ij}-A_j)}{[(\text{Variance }(\Sigma_i C_{ij}X_{ij}-A_j))]^{\frac{1}{2}}}$$

Assumption 3

Similarly, whatever may be the distribution of $\Sigma_i D_{ij}X_{ij}$, let G(.) denote the cumulative distribution function of

$$\frac{(\Sigma_i D_{ij}X_{ij}-B_j)-E(\Sigma_i D_{ij}X_{ij}-B_j)}{[(\text{Variance }(\Sigma_i D_{ij}X_{ij}-B_j))]^{\frac{1}{2}}}$$

Step 3

Transform stochastic constraints to deterministic constraint. Consider, constraint 1.

$$\text{Prob}\{\Sigma^n{}_{i=1}C_{ij}X_{ij}\geq A_j\}\geq\alpha_j \text{ for any fixed } j$$

Let,

$$\Sigma^n{}_{i=1}C_{ij}X_{ij}=Z_j\cdots\cdots \tag{4.5}$$

Then,

$$E(Z_j)=E(\Sigma^n{}_{i=1}C_{ij}X_{ij})=\mu_{1j}X_{1j}+\mu_{2j}X_{2j}+\cdots\cdots+\mu_{nj}X_{nj}=\mu_j \tag{4.6}$$

$$\text{Variance}(Z_j)=\text{Variance }(\Sigma_i C_{ij}X_{ij})=\sigma^2{}_{1j}X^2{}_{1j}+\sigma^2{}_{2j}X^2{}_{2j}+\cdots\cdots+\sigma^2{}_{nj}X^2{}_{nj} \tag{4.7}$$

Under the assumption variance (C_{ij}) are independently distributed. Substituting (4.6) and (4.7) in (4.5) we get:

$$\Pr\{Z_j\geq A_j\}\geq\alpha_j \text{ for a fixed } j \tag{4.8}$$

$$1-\Pr\{Z_j\leq A_j\}\leq A_j\geq\alpha_j \text{ for a fixed } j \tag{4.9}$$

$$\Pr\{Z_j\leq A_j\}\leq 1-\alpha_j \text{ for a fixed } j \tag{4.10}$$

Using results of the theorems 4.1 and 4.2 we obtain:

$$=\int_{-\infty}^{A_j} \Pr(Z_j,\ \mu_j,\ \Sigma_j)dz_j \leq 1-\alpha_j \text{ for a fixed } j \tag{4.11}$$

$$=\int_{-\infty}^{\frac{A_j-\mu_j}{\Sigma_j}} \Pr(Z_j,\ 0,\ 1)dz_j \leq 1-\alpha_j \text{ for a fixed } j \tag{4.12}$$

$$=F\left[\frac{A_j-\mu_j}{\Sigma_j}\right] \leq 1-\alpha_j \text{ for a fixed } j \tag{4.13}$$

$$=\frac{A_j-\mu_j}{\Sigma_j} \leq F^{-1}(1-\alpha_j) \tag{4.14}$$

$$=A_j \leq \mu_j+\Sigma_j F^{-1}(1-\alpha_j) \tag{4.15}$$

Substituting the values of μ_j and Σ_j in (4.15) we get

$$=A_j \leq \mu_{1j}X_{1j}+\mu_{2j}X_{2j}+\cdots\cdots+\mu_{nj}X_{nj} + F^{-1}(1-\alpha_j)\sqrt{\sigma^2{}_{1j}X^2{}_{1j}+\sigma^2{}_{2j}X^2{}_{2j}+\cdots\cdots+\sigma^2{}_{nj}X^2{}_{nj}} \tag{4.16}$$

Consider constraint 2,

$$\text{Prob}\{\Sigma_i D_{1j}X_{1j} \leq B_j\} \geq \beta_j \text{ for a fixed } j \tag{4.17}$$

Let

$$\Sigma_i D_{ij}X_{ij}=Y_j \tag{4.18}$$

Then

$$E(Y_j)=E(\Sigma i D_{ij}X_{ij})=\gamma_{1j}X_{1j}+\gamma_{2j}X_{2j}+\cdots\cdots+\gamma_{nj}X_{nj}=\Gamma_j \tag{4.19}$$

$$\text{Variance }(Y_j)=\text{Variance }(\Sigma_i D_{ij}X_{ij})$$

$$=X^2{}_{1j}W^2{}_{1j}+X^2{}_{2j}W^2{}_{2j}+\cdots\cdots+X^2{}_{nj}W^2{}_{nj}=\Omega^2{}_j \tag{4.20}$$

Under the assumption that the variables (D_{ij}) are independently distributed. Substituting (4.18) and (4.19) we get,

$$\Pr(Y_j \leq B_j) \geq \beta_j \cdots\cdots \tag{4.21}$$

Using the theorems 4.1 and 4.2 we get,

$$\int_{-\infty}^{Bj} \Pr(Y_j, \Gamma_j, \Omega^2{}_j)dY_j \text{ for a fixed } j \tag{4.22}$$

$$=\int_{-\infty}^{\frac{B_j-\Gamma_j}{\Omega_j}} \Pr(Y_j,\ 0,\ 1)dY_j \text{ for a fixed } j \tag{4.23}$$

$$G\left[\frac{B_j-\Gamma_j}{\Omega_j}\right] \geq \beta_j$$

$$B_j \geq \Gamma_j+G^{-1}(\beta_j)\Omega_j \tag{4.24}$$

Substituting the values of Γ_j and Ω_j we get

$$B_j \geq \gamma_{1j}X_{1j}+\gamma_{2j}X_{2j}+\cdots\cdots+\gamma_{nj}X_{nj}+G^{-1}(\beta_j)\sqrt{X^2{}_{1j}W^2{}_{1j}+X^2{}_{2j}W^2{}_{2j}+\cdots\cdots+X^2{}_{nj}W^2{}_{nj}}$$

Consider constaint (3).

$$\Sigma^n{}_{i=1}X_{ij} \leq N_j-S_j+\Sigma^n{}_{i=1}R_{i,j-1}X_{ij-1}+\Sigma^n{}_{i=1}X_{i,j-1} \text{ for a fixed } j$$

This is already in a deterministic form. So no reduction is required. Consider constraint (4).

It is a deterministic non-negative constraint and so no reduction is required. Without loss of generality let:

$$\Sigma^n{}_{i=1}R_{ij-1}X_{ij-1}=0_{ij-1}$$

and

$$\Sigma^n{}_{i=1}X_{ij-1}=M_{j-1}$$

Denote the reduced problem as 4.2.

Problem 4.2

Maximize $E\ [\Sigma^n{}_{i=1}C_{ij}X_{ij}-\Sigma^n{}_{i=1}D_{ij}X_{ij}]$

Subject to:

Constraint (1) $\Sigma^n{}_{i=1}\mu_{ij}X_{ij}-A_j+F^{-1}(1-\alpha_j)\sqrt{\Sigma^n{}_{i=1}\sigma^2{}_{ij}X^2{}_{ij}}\geq 0$

Constraint (2) $B_j-\Sigma^n{}_{i=1}\gamma_{ij}X_{ij}-G^{-1}(\beta_j)\sqrt{\Sigma^n{}_{i=1}W^2{}_{ij}X^2{}_{ij}}\geq 0$

Constraint (3) $N_j-S_j-\Sigma^n{}_{i=1}X_{ij}+O_{j-1}+M_{j-1}\geq 0$

Constraint (4) $X_{ij}\geq 0$

SECTION V : PROPERTIES OF THE REDUCED MODEL

Property 1

Constraint (1) namely,

$$(\Sigma^n{}_{i=1}\mu_{ij}X_{ij}-A_j)+F^{-1}(1-\alpha_j)\sqrt{\Sigma X^2{}_{ij}\sigma^2{}_{ij}}$$

is convex provided $\alpha_j\geq .5$.

Proof

First we shall prove the convexity of

$$\sqrt{\Sigma^n{}_{i=1}X^2{}_{ij}\sigma^2{}_{ij}}$$

Dropping the suffix j we obtain,

$$\sqrt{\Sigma_i X^2{}_i\sigma^2{}_i}=(X^1VX)^{\frac{1}{2}} \quad (5.1)$$

Where V is a positive semi-definite or definite matrix[3] and X, Y, Z be the vectors that satisfy the following relation:

$$Z=\lambda X+(1-\lambda)Y \qquad 0\leq\lambda\leq 1$$

In order to prove convexity of $(X^1VX)^{\frac{1}{2}}$, we have to show that

$$\text{sign of } \{\sqrt{Z^1VZ}-[\lambda\sqrt{X^1VX}+(1-\lambda)\sqrt{Y^1VY}]\} \quad (5.2)$$

is non-positive. Consider the

$$\text{sign of } \{\sqrt{Z^1VZ}-[\lambda\sqrt{X^1VX}+(1-\lambda)\sqrt{Y^1VY}]\}$$

$$\text{sign of } \{(Z^1VZ)-[\lambda\sqrt{X^1VX}+(1-\lambda)\sqrt{Y^1VY}{}^2]\} \quad (5.3)$$

since sign $(a^2-b^2)=$sign $(a-b)$ for any a and b

$$=\text{sign } \{\lambda^2X^1VX+2\lambda(1-\lambda)X^1VY+(1-\lambda)^2Y^1VY$$
$$-\lambda^2X^1VX-2\lambda(1-\lambda)\sqrt{X^1VX}\sqrt{Y^1VY}-(1-\lambda)^2Y^1VY\}$$

$$=\text{sign } \{2\lambda(1-\lambda)(X^1VY)-\sqrt{X^1VX}\sqrt{Y^1VY}\} \quad (5.4)$$

Since the matrix V is a positive semi-definite for any arbitrary number t we have

$$(tX+Y)^1V(tX+Y)=t^2X^1VX+2t^1XVY+Y^1VY\geq 0 \quad (5.5)$$

Hence

$$(X^1VY)^2\leq(X^1VX)(Y^1VY)$$

or

3) The variance covariance matrix is always either positive definite or positive semi-definite.

$$X^1VY \leq \sqrt{X^1VX}\ \sqrt{Y^1VY}$$

since X^1VX and Y^1VY are non-negative, so

$$[X^1VY - \sqrt{X^1VX}\ \sqrt{Y^1VY}] \leq 0 \tag{5.6}$$

Substituting in sign $\{2\lambda(1-\lambda)$(negative quantity)$\}$ which is non-positive since $\lambda \geq 0$ and $\lambda \leq 1$, it follows that $(X^1VX)^{\frac{1}{2}}$ is convex. $(\Sigma_i \mu_i X_i - A)$ is a linear function in X and is always convex [16]. Thus the first term is convex and so is $(X^1VX)^{\frac{1}{2}}$.

The expression will be convex provided $F^{-1}(1-\alpha_j)$ is always greater than or equal to 0, since a positive linear combination of convex function is always convex. We require

$$F^{-1}(1-\alpha_j) \geq 0 \tag{5.7}$$

or

$$F(0) \geq 1-\alpha_j \tag{5.8}$$

or

$$\alpha_j \geq .5 \tag{5.9}$$

when the distribution is normal [14].

Property 3

Constraint (3) namely,

$$\{[\Sigma_i \gamma_{ij} X_{ij} + B_j] - G^{-1}(\beta_j) \sqrt{\Sigma_i X^2_{ij} W^2_{ij}}\} \tag{5.10}$$

is convex provided $G^{-1}(\beta_j) \leq 0$ or $\beta_j \leq .5$.

Proof

$\sqrt{\Sigma_i X^2_{ij} W^2_{ij}}$ is convex since the variance covariance matrix W is always either positive semidefinite or positive definite. For proof refer to Property 1.

$(-\Sigma_i \gamma_{ij} X_{ij} + B_j)$ is a linear function in X and hence convex. Thus the expression (5.10) will be convex provided $G^{-1}(\beta_j) = 0$ since a positive linear combination of convex function is always convex.

For example, we require

$$G^{-1}(\beta_j) \leq 0 \tag{5.11}$$

or

$$G(0) \leq \beta_j \tag{5.12}$$

or

$$\beta_j \leq .5 \tag{5.13}$$

when distribution is normal. Constraint (3) is linear in X and so convexity is automatically obtained. Constraint (4) is a non-negativity requirement.

Property 2

Problem 4.2 is a Convex programming problem. By definition a convex programming problem is maximizing (minimizing) a concave (convex) function defined over a convex set.

The objective function Max $E\{\Sigma^n_{i=1}(C_{ij} - D_{ij})X_{ij}\}$ is concave since it is linear in X.

The constraints (1)—(4) are shown to be convex. Thus the problem is a Convex programming problem.

Theorem 5.1

If the objective function is concave defined over a convex set, then the problem has all local solutions global. (For simplistic reasons denote f(.) as the objective function and S as the constraint set.)

For example:

$$\text{Max } f(X)$$

Subject to:

$$X \in S$$

Proof

We shall prove by contradiction. Let us assume that there is a local solution which is not global, X_e and let the global maximum be attained at X_g then,

$$f(X_g) > f(X_e)$$

By convexity $\lambda X_e + (1-\lambda)X_g \in S$ for all $0 \leq \lambda \leq 1$.

Using the fact that $f(.)$ is concave we have,

$$f[\lambda X_e + (1-\lambda)X_g] \geq \lambda f(X_e) + (1-\lambda)f(X_g) > \lambda f(X_e) + (1-\lambda)f(X_e) = f(X_e)$$

For λ near 1 the local solution contradicts the local maximality. Hence all local solutions are global.

CONCLUSION

The deterministic model has the structure of a non-linear programming problem with a number of unknown input parameters. All these parameters can either be computed directly from the Wall Street Journal or in the absence of information, evaluated using a Delphi panel of experts (15). Since the problem has the property of convexity, a number of algorithms exist in the literature to determine the optimal solution. To test the efficacy of the model we developed a computer program which is based on the Barrier method for non-linear programming model and applied to a randomly seleced set of stocks. The output of the program clearly brought out the usefulness and capabilities of the model (11). Further, this model is a generalized one in the sense that it contains all the factors which are considered in the existing literature as being germane to effective decision making. So, many existing models in the area of portfolio selection problem can be shown as a subcase of the present model using a taxonomical scheme (11).

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