Number of Equivalence Classes of a Parallel Flats Fraction for the 3^n Factorial Design

Jung-Koog Um*

ABSTRACT

A parallel flats fraction for the 3^n factorial experiment is symbolically written as $A\underline{t}=C=(\underline{C}_1 \underline{C}_2 \cdots \underline{C}_f)$ where C is a rxf matrix and A is rxn matrix with rank r. It is shown that the set of all possible parallel flats fraction C for a given A and given size can be partitioned into equivalence classes. The number of those classes are enumerated in general.

1. Introduction

A parallel flats fraction for the 3^n factorial experiment is defined as the union of flats, $\{\underline{t} \mid A\underline{t}=\underline{C}_i (\text{mod } 3),\ i=1,2,\cdots f\}$ and is symbolically written as $A\underline{t}=C=(\underline{C}_1\underline{C}_2\cdots\underline{C}_f)$ where A is a $r\times n$ matrix with rank r and C is a $r\times f$ matrix. Note that f denotes the number of flats.

It is important to relate the solution $\underline{t}$ to the C-matrix. Since C is a $r\times f$ matrix and all entries are elements of GF(3), there are $3^{r\times f}$ different matrices for C. If attention is restricted to the different columns then there are $\binom{3^r}{f}\times(f!)$ possible matrices.

2. Basic Theorems

We quote the following two theorems derived by Anderson and Mardekian (1979).

Theorem 1. Let T be the design obtained from the 3^n parallel flats fraction given by the solutions of $A\underline{t}=C$ and let T^* be the design obtained from the solutions to $A\underline{t}=C^{**}$ where C^{**} is obtained by adding the vector V with components in GF(3) to each of the columns of C. Then

(1) E is estimable from the runs of T if and only if
E is estimable from the runs of T^*;

(2) if X and X^* are the X-matrices corresponding
to T and T^* then $\det(X^{*\prime} X^*)=\det(X' X)$.

Theorem 2. Let T be the design obtained from the 3^n parallel flats fraction given by

* Department of Computer Science, Sogang University

the solutions of $A\underline{t}=C$ and let T^* be the design obtained from the solutions of $A\underline{t}=2C$. Then

(1) E is estimable from the runs of T if and only if
E is estimable from the runs of T^*;

(2) if X and X^* are the X-matrices corresponding to
T and T^* then det $(X^{*\prime}\ X^*)=$det $(X'\ X)$.

Therem 1 and Theorem 2 can be combined to establish designs which are equivalent with respect to estimability and the determinant of the resulting $X'\ X$-matrices. In particular, If T is obtained from solutions to $A\underline{t}=C$ and T^* is obtained from solutions to $A\underline{t}=2\ C+(\underline{V},\ \underline{V},\cdots,\underline{V})$ then the designs T and T^* can be considered equivalent.

An implication of Theorem 1 is that no generality is lost with respect to estimability and the determinant of the information matrix if attention is restricted to parallel-flats fractions where the first column of C is chosen to be the vector of O's$\in GF(3)$ in order to define the observations in the first flat. Inthis case there are $\binom{3^r-1}{f-1}\times(f-1)!$ different ways to obtain C-matrix.

We choose $\underline{V}$ such that there exists one column of O's after adding $\underline{V}$ to each column of C. Then the column with $\underline{O}$'s will be the first column of the matrix which is obtained by adding $\underline{V}$ to C.

Example. Let $C=\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}$ and $\underline{V}=\begin{bmatrix}0\\2\end{bmatrix}$ Then

$$C+\underline{V}=\begin{bmatrix}0&0&1\\2&0&2\end{bmatrix}\rightarrow\begin{bmatrix}0&0&1\\0&2&2\end{bmatrix}$$

Another implication of Theorem 1 and Theorem 2 is that C and C^*, where C^* is obtained by adding the vector $\underline{V}$ with components in $GF(3)$ to each of the columns of C, are equivalent, and also C and $2C$ are equivalent. That is, C, C^* and $2C$ belong to the same equivalence class.

Theorem 3. Suppose that C is a $r\times f$ matrix such that the first column contains only O's. Then the maximum number of elements in an equivalence class is $2\times f\times(f-1)!$.

Proof. For each column of C, except the first column with O's, there exists a nonzero vector $\underline{v}$ such that makes that column $\underline{O}$ after adding the $\underline{V}$. Therfore, there are $f-1$ matrices and hence f matrices (including C itself) belonging to the same equivalence class.

It is clear that no generality is lost with respect to estimabillity and the determinant of the information matrix when the columns of C are permuted, except for the first column with O's. Therefore, for each matrix of f matrices, $(f-1)!$ matrices belong to the same equivalence class.

The matrices obtained by multiplying these $f\times(f-1)!$ matrices by 2 belong to the same equivalence class. This completes the proof.

If C is a 2×3 matrix then there are 56 matrices and the maximum number of elements in an equivalence class is 12.

3. Number of Equivalence Classes of C-matrix

In order to find the number of equivalence classes we apply Burnside's Lemma. Let G be a group of permutations acting on a finite set S. G induces an equivalence relation on S. Two elements s_1, $s_2 \in S$ are equivalent, $s_1 \sim s_2$, if and only if there exists $\sigma \in G$ such that $(s_1)\ \sigma = s_2$. This equivalence relation partitions S into equivalence classes.

Lemma. Burnside

The number of equivalence classes of S equals $|G|^{-1}\sum_{\sigma\in G}\chi_s(\sigma)$, where $|G|$ denotes the number of elements of G, and for each $\sigma\in G, \chi_s(\sigma)$ denotes the number of elements of S that are invariant under σ, that is, $\chi_s(\sigma)=|\{s\in S\,|\,(s)\sigma=s\}|$.

Proof. We consider all pairs $(\sigma,\ s)$ with $\sigma\in G$, $s\in S$, and $(s)\sigma=s$. The number of these pairs can be counted in two ways. For each fixed $\sigma\in G$ we can count the number of s satisfying $(s)\sigma=s$ and, therefore, the number of pairs is $\sum_{\sigma\in G}\chi_s(\sigma)$. On the other hand, for each $s\in S$ we can count the number of $\sigma\in G$ with $(s)\sigma=s$. Donoting this number by $|G_s|$, we have

$$\sum_{\sigma\in G}\chi_s=(\sigma)=\sum_{s\in S}|G_s|$$

Let C_s be the set of equivalence classes and let s_1 and s_2 belong to the same equivalence class O. Then $|G_{s_1}|=|G|/|0|=|G_{s_2}|$ Therefore,

$$\sum_{\sigma\in G}\chi_s(\sigma)=\sum_{s\in S}|G_s|=\sum_{0\in C_S}\ \sum_{s\in 0}|G_s|$$

$$=\sum_{0\in C_S}|0|\cdot|G|/|0|=|G|\cdot|C_s|\cdot$$

Hence the number of equivalence classes is

$$|C_S|=|G|^{-1}\sum_{\sigma\in G}\chi_s(\sigma).$$

This completer the proof.

We now apply Burnside's Lemma to find the number of equivalence classes for C-matrix. Suppose that C-matrix has r rows and f columns, and all entries are elements of $GF(3)$. Our attention is restricted to distinct columns only.

Let S_f be the permutation group on the set $\{0,1,2,\cdots,f-1\}$ and V be the set of all possible columns for the $r\times f$ matrix. Let $W=\{1,2\}$ and S be the set of all possible $r\times f$ matrix which has different columns.

Then clearly $|S_f|=f!\,|V|=3^r$, $|W|=2$, and $|S|=\dfrac{|V|!}{(|V|-f)!}$

Let $\underline{u}_0$, $\underline{u}_1$, $\underline{u}_2,\cdots,\underline{u}_{f-1}$ be the column vectors of $r\times f$ matrix and each permutation of S_f acts on the columns of the matrix. For example, if we have $(012)\in S_3$, then

$$(\underline{u}_0\ \underline{u}_1\ \underline{u}_2)\ (012)=(\underline{u}_2\ \underline{u}_0\ \underline{u}_1).$$

For $v \in V$ and $w \in W$ let us define the following operations

$$(\underline{u}_0\ \underline{u}_1 \cdots \underline{u}_{f-1})v = (\underline{u}_0 + v \cdots \underline{u}_{f-1} + v)$$

$$(\underline{u}_0\ \underline{u}_1 \cdots \underline{u}_{f-1})w = (\underline{u}_0 \times w\ \ \underline{u}_1 \times w \cdots \underline{u}_{f-1} \times w)$$

Then clearly we have the following relations

(1) $v \circ \sigma = \sigma \circ v$

(2) $\sigma \circ w = w \circ \sigma$

(3) $v \circ w = w \circ (v \times w)$. If $w = 2$, then $v \circ w = w \circ (-v)$.

Let G be the set of all $\sigma \circ v \circ w$, $\sigma \in S_f$, $v \in V$ and $w \in W$, that is, $G = \{\sigma \circ v \circ w \mid \sigma \in S_f, v \in V, w \in W\}$. Define the binary operation $\triangle$ by

$$\begin{aligned}(\sigma_1 \circ v_1 \circ w_1) \triangle (\sigma_2 \circ v_2 \circ w_2) &= (\sigma_1\sigma_2) \circ (v_1 + v_2) \circ 1, \text{ if } w_1 = w_2 = 1 \\ &= (\sigma_1\sigma_2) \circ (v_1 + v_2) \circ w_2, \text{ if } w_1 = 1,\ w_2 = 2 \\ &= (\sigma_1\sigma_2) \circ v_3 \circ w_1, \text{ where } v_1 \times w_1 + v_2 = v_3 \circ w_1, \\ &\qquad \text{if } w_1 = 2,\ w_2 = 1 \\ &= (\sigma_1\sigma_2) \circ v_3 \circ 1, \text{ where } v_3 = (v_1 \times w_1 + v_2) \times w_2, \\ &\qquad \text{if } w_1 = w_2 = 2.\end{aligned}$$

Then clearly $e = e \circ \underline{O} \circ 1$ is an identity element in G and every element in G has an inverse element in G. It can be easily shown that

$$((\sigma_1 \circ v_1 \circ w_1) \triangle (\sigma_2 \circ v_2 \circ w_2)) \triangle (\sigma_3 \circ v_3 \circ w_3) = (\sigma_1 \circ v_1 \circ w_1) \triangle ((\sigma_2 \circ v_2 \circ w_2) \triangle (\sigma_3 \circ v_3 \circ w_3))$$

Hence G is a group with $|G| = f! \times |V| \times 2$.

Let two matrices c_1, $c_2 \in s$ be equivalent, $c_1 \sim c_2$, if and only if, there exists $\sigma \circ v \circ w \in G$ such that

$$(c_1)(\sigma \circ v \circ w) = c_2.$$

This equivalence relation partitions S into equivalence classes. Hence we set forth the following theorem.

Theorem 4. The number of equivalence classes of S which is the set of $r \times f$ matrices with distinct columns equals

$$\begin{aligned}|G|^{-1} \sum_{\sigma \in G} \chi_s(\sigma) &= |G|^{-1}(|S| + H_2K_2) \qquad \text{if } 3 \nmid f \\ &= |G|^{-1}(|S| + H_2K_2 + H_1K_1) \text{ if } 3 \mid f\end{aligned}$$

where $|V| = 3^r$, $|G| = f! \cdot |V| \cdot 2$, $S = \dfrac{|V|!}{(|V| - f)!}$

$$H_1 = |V|(|V| - 3) \cdots (|V| - (f-3)) \text{ if } 3 \mid f$$

$$K_1 = \frac{f!(|V| - 1)}{3^{(f/3)} \cdot ((f/3)!)} \text{ if } 3 \mid f$$

$$\begin{aligned}H_2 &= (|V| - 1)(|V| - 2) \cdots (|V|) - (f-2)) \text{ if } f \text{ is odd} \\ &= (|V| - 1)(|V| - 2) \cdots (|V| - (f-1)) \text{ if } f \text{ is even}\end{aligned}$$

$$K_2 = \frac{f!\,|V|}{2^{[f/2]} \cdot [f/2]!}$$

where $[f/2]$ denotes the largest integer which is smaller than $f/2$.

The details of the proof are still somewhat lengthy and have been omitted here.

Table 1 shows the number of equivalence classes for the various values for r and f.

Table 2 shows the equivalence classes of C-matrix for the 3^4 factorial, and matrics with the first column containing $\underline{O}$ are presented.

Table 1 The Number of Equivalence Classes of C-Matrix

r \ f	2	3	4	5	6	7	8
2	4	8	10	10	8	4	1
3	13	65	364	1, 534	5, 642	16, 588	41, 470
4	40	560	10, 660	158, 548	2, 008, 448	21, 469, 240	198, 590, 470
5	121	4, 961	295, 240	13, 942, 588	553, 057, 604	18, 720, 164, 584	522, 244, 855, 228
6	364	44, 408	8, 038, 030	1, 160, 757, 598	140, 643, 544, 860	43, 578, 166, 217, 508	386, 973, 008, 116, 223

Table 2 Equivalence Classes of C-Matrix For The 3^4 Factorial

Class 1

001	001	022	010	010	022
012	021	021	021	012	012
002	002	011	020	020	011
021	012	012	012	021	021

Class 2

012	021	021	021	012	012
001	001	022	010	010	022
021	012	012	012	021	021
002	002	011	020	020	011

Class 3

001	001	022	010	010	022
010	022	010	001	022	001
002	002	011	020	020	011
020	011	020	002	011	002

Class 4

001	001	022	010	010	022
011	020	002	011	002	020
002	002	011	020	020	011
022	010	001	022	001	010

Class 5

000	000

012	021
Class 6	
012	021
000	000
Class 7	
012	021
012	021
Class 8	
012	021
021	012

References

(1) DeBruijn, N.G. (1964) Pólyás theory of counting, Applied Combinatorial Mathematics, Wiley.

(2) Kerber, A. (1970) Representations of permutation groups, Springer-Verlag

(3) Anderson, D.A. and Mardekian, J. (1979) Paraller flats factions for the 3^n factorial experiment. The Institute of Mathematical Statistics Bulletin, 8, 223

(4) Srivastava, J.N. Notes on parallel flats fractions, Unpublished.

(5) Srivastava, J.N. and Ghosh, S (1977) Balanced 2^m factorial designs of resolution V which allow search and estimation of one extra unknown effect, $4 \leq m \leq 8$. Communications in Statistics, $A6$ (2), 141—166