A Lattice Distribution†

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Abstract

It is shown that a lattice distribution defined on a set of n lattice points $L(n, \delta)=\{\delta, \delta+1, \cdots, \delta+n-1\}$ is a distribution induced from the distribution of convolution of independently and identically distributed ($i.i.d.$) uniform [0, 1] radom variables. Also the m-th moment of the lattice distribution is obtained in a quite different approach from Park and Chung (1978). It is verified that the distribution of the sum of n $i.i.d.$ uniform [0, 1] random variables is completely determined by the lattice distribution on $L(n, \delta)$ and the uniform distribution on [0, 1]. The factorial moment generating function, factorial moments, and moments are also obtained.

1. Introduction

Let $X_1, X_2, \cdots, X_n$ be $i.i.d.$ uniform [0, 1] random variables and let $f_n(x)$ denote the probability density function ($p.d.f.$) of

$$S_n=\sum_{i=1}^{n} X_i \tag{1.1}$$

For a given $\delta: 0\leq\delta\leq1$, let $L(n, \delta)$ denote a set of lattice points defined by

$$L(n, \delta)=\{\delta, \delta+1, \cdots, \delta+n-1\}. \tag{1.2}$$

Consider a function $f_n(x\,;\delta)$ given by

$$f_n(x\,;\delta)=\begin{cases} f_n(x) & \text{if } x \in L(n, \delta) \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Later, it will be proved that $f_n(x\,;\delta)$ is a probability function, and for this reason, the distribution associated with $f_n(x\,;\delta)$ is called the "lattice distribution" on the set $L(n, \delta)$.

In this paper the properties of the lattice distribution are studied. In section 2, it is shown that the lattice distribution can be naturally induced from the distribution of S_n, as the conditional distribution of S_n, given the fractional part of S_n. To do this, we show that $f_n(x\,;\delta)$ defined by (1.3) is, in fact, a probability function of the lattice distribution. Section 3 deals with the moments of the lattice distribution.

2. The Derivation of the Lattice Distribution

Lamma 2.1. Let $S(\delta, r, n)$ be defined by

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$$S(\delta, r, n)=\frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(\delta+n-i)^r, \qquad r, n=1, 2, \cdots \tag{2.1}$$

for any real number δ. Then, we have

$$S(\delta, r, n)=\begin{cases}\sum_{j=0}^{r-n}\delta^j\binom{r}{j}S(0, r-j, n) & \text{if } r>n\\ 1 & \text{if } r=n\\ 0 & \text{if } r<n.\end{cases}$$

Proof. First, note that, for any real number t,

$$\begin{aligned}(e^t-1)^n&=\sum_{j=0}^{n}(-1)^j\binom{n}{j}e^{(n-j)t}\\ &=\sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-j)^k\\ &=\sum_{k=0}^{\infty}\frac{n!}{k!}S(0, k, n)t^k.\end{aligned}$$

Hence, we have

$$\begin{cases}(e^t-1)^n=\sum_{k=n}^{\infty}\frac{n!}{k!}S(0, k, n)t^k\\ S(0, r, n)=\begin{cases}1 & \text{if } r=n\\ 0 & \text{if } r<n.\end{cases}\end{cases} \tag{2.2}$$

Now, consider the following relation :

$$\begin{aligned}e^{\delta t}(e^t-1)^n&=\sum_{k=0}^{\infty}\sum_{r=n}^{\infty}\frac{n!}{k!r!}S(0, r, n)\delta^k t^{k+r}\\ &=\sum_{j=n}^{\infty}\frac{t^j}{j!}\left[\sum_{i=0}^{j-n}\frac{j!\delta^i}{i!(j-i)!}n!S(0, j-i, n)\right]\\ &=\sum_{j=n}^{\infty}\frac{t^j}{j!}n!\left[\sum_{i=0}^{j-n}\delta^i\binom{j}{i}S(0, j-i, n)\right]\end{aligned} \tag{2.3}$$

On the other hand,

$$\begin{aligned}e^{\delta t}(e^t-1)^n&=\sum_{j=0}^{n}(-1)^j\binom{n}{j}e^{(\delta+n-j)t}\\ &=\sum_{k=0}^{\infty}\frac{t^k}{k!}n!S(\delta, k, n)\end{aligned} \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$S(\delta, r, n)=\begin{cases}\sum_{j=0}^{r-n}\delta^j\binom{r}{j}S(0, r-j, n) & \text{if } r>n\\ 1 & \text{if } r=n\\ 0 & \text{otherwise.}\end{cases}$$

Note that Lemma 2.1 generalizes the results for $\delta=0$ in W. Feller (1968), and that $S(0, r, n)$ is the stirling number of the second kind. The following result is also needed, (see, S.S. Wilks (1962), for example).

Lemma 2.2. For $n=2, 3, \cdots$, let $f_n(x)$ be the *p. d. f.* of the random variable defined by (1.1). Then, $f_n(x)$ is given by

$$f_n(x)=\begin{cases}\dfrac{1}{(n-1)!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i)_+^{\,n-1} & \text{for } 0\le x\le n\\ 0 & \text{otherwise,}\end{cases} \tag{2.5}$$

where

$$x_+=\begin{cases}x & \text{if } x\ge 0\\ 0 & \text{if } x<0.\end{cases}$$

Now, let $W_n=[S_n]$ and $Y_n=S_n-W_n$ be the integer part and the fractional part of S_n, respectively.

Then, the *p.d.f.* $g_n(y)$ of Y_n is given by

$$g_n(y)=\begin{cases}\sum_{j=0}^{n-1}f_n(y+j) & \text{if } 0\le y\le 1\\ 0 & \text{otherwise.}\end{cases} \tag{2.6}$$

In the sequel, it will be shown that Y_n is in fact a uniform random variable on [0, 1]. To this end, we need the following results.

Lemma 2.3. Let $S(\delta, r, n)$ be the generalized stirling number of the second kind defined by (2.1). Then, the following identity holds.

$$n!S(\delta,r,n)=\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^i\binom{n+1}{i}(\delta+j-i)^r \tag{2.7}$$

Proof. The lemma can be proved by the following sequence of identities :

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^i\binom{n+1}{i}(\delta+j-i)^r\\ &=\sum_{q=0}^{r}\binom{r}{q}\delta^q\sum_{l=0}^{n}(-1)^l\binom{n}{l}(n-l)^{r-q}\\ &=\sum_{q=0}^{r}\binom{r}{q}\delta^q n!S(0,r-q,n)\end{aligned}$$

Hence, the result follows from Lemma 2.1.

Now, the main result in this section is given in the next theorem.

Theorm 2.1. For any fixed δ : $0\le\delta\le 1$,

$$\sum_{j=0}^{n-1}f_n(j+\delta)=1.$$

Therefore, $f_n(x:\delta)$ can be considered as a probability function on the set $L(n,\delta)$.

Proof. It follows from Lemmas 2.1, 2.2 and 2.3 that

$$\begin{aligned}\sum_{j=0}^{n-1}f_n(j+\delta)&=\frac{1}{(n-1)!}\sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^i\binom{n}{i}(\delta+j-i)^{n-1}\\ &=S(\delta,n-1,n-1)\\ &=1.\end{aligned}$$

Thus the theorem follows.

It can be easily shown that, for $j=0,1,\cdots,n-1$ and δ : $0\le\delta\le 1$,

$$P_r[W_n=j,\ Y_n\le\delta]=\int_0^\delta f_n(j+y)dy.$$

Hence, the results related to the distributions of W_n and Y_n can be summarized as follows :

Corollary 2.1. Let $Y_n=S_n-[S_n]$ and $W_n=[S_n]$ be the decimal fractional part and the integer part of the random variable S_n defined by (1.1), respectively. Then,

(a) $f_n(j+\delta)$, as a function of $j=0, 1, \cdots, n-1$ and $\delta : 0\leq\delta\leq 1$, is the joint $p.d.f.$ of W_n and Y_n,

(b) the marginal distribution of Y_n is a uniform distribution on $[0, 1]$

(c) $f_n(j+\delta)$, as a function of j only, is the conditional probaility function of W_n, given $Y_n=\delta$.

Remark: Theabove results are interesting in the sense that Y_n is uniformly distributed on $[0, 1]$ as we might expect, and that Y_n and W_n are dependent contrary to our intuition.

The distribution on the set $L(n, \delta)$ associated with $f_n(x : \delta)$ defined by (1.3) is clearly the conditional distribution $W_n+\delta$, given $Y_n=\delta$. Such a distribution will be called the "lattice distribution" on the set $L(n, \delta)$. Note that the distribution of S_n is completely determined by the lattice distribution on $L(n, \delta)$ and the uniform distribution on $[0, 1]$. Hence it is worthwhile to investigate the properties of the lattice distribution.

3. Moments of Lattice Distribution.

Lemma 3.1. The $f.m.g.$ function $\phi_n(t\ ;\ \delta)$ of the lattice distribution on the set $L(n, \delta)$ can be written as

$$\phi_n(t\ ;\ \delta)=\frac{(1+t)^{\delta}}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}\delta^k\Big[\sum_{j=0}^{n-1}(-t)^j(1+t)^{n-1-j}\ (n-1-j)!\ S(0, n-1-k, n-1-j)\Big] \tag{3.1}$$

Proof. It follows from Lemma 2.2 that the $f.m.g.$ function $\phi_n(t\ ;\ \delta)$ is given by

$$(n-1)!\phi_n(t\ ;\ \delta)=\sum_{j=0}^{n-1}(1+t)^{\delta+j}\sum_{i=0}^{j}(-1)^i\binom{n}{i}(\delta+j-1)^{n-1}$$

Therefore, $\phi_n(t\ ;\ \delta)$ can be obtained as the coefficient of u^{n-1} in the series expansion of the following function :

$$\begin{aligned}
&\sum_{j=0}^{n-1}(1+t)^{\delta+j}\sum_{i=0}^{j}(-1)^i\binom{n}{i}e^{u(\delta+j-i)}\\
&=(1+t)^{\delta}e^{u\delta}\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}(-1)^i\binom{n}{i}(1+t)^je^{(j-i)u}\\
&=\frac{(1+t)^{\delta}e^{u\delta}}{1-(1+t)e^u}\Big[\sum_{i=0}^{n-1}\binom{n}{i}(-1)^i(1+t)^i-\sum_{i=0}^{n-1}\binom{n}{i}(-1)^i(1+t)^i\{(1+t)e^u\}^{n-i}\Big]\\
&=(1+t)^{\delta}e^{u\delta}\sum_{j=0}^{n-1}(-t)^j(1+t)^{n-1-j}(e^u-1)^{n-1-j}\\
&=(1+t)^{\delta}\sum_{j=0}^{n-1}(-t)^j(1+t)^{n-1-j}\sum_{k=0}^{\infty}\frac{u^k\delta^k}{k!}\sum_{l=0}^{\infty}\frac{u^l}{l!}(n-1-j)!\ S(0, l, n-1-j)\\
&=(1+t)^{\delta}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n-1}(-t)^j(1+t)^{n-1-j}(n-1-j)!\ S(0, l, n-1-j)\delta^ku^{k+l}/(k!l!)
\end{aligned}$$

Hence, $$(n-1)!\phi_n(t\ ;\ \delta)=(1+t)^{\delta}\sum_{k=0}^{n-1}\binom{n-1}{k}\delta^k\Big[\sum_{j=0}^{n-1}(-t)^j(1+t)^{n-1-j}\ (n-1-j)!\ S(0, n-1-k, n-1-j)\Big]$$

Lemma 3. 2. The m-th ($m=1, 2, \cdots$) factorial moment $\mu_{n,\delta}^{(m)}$ of the lattice distribution on the set $L(n, \delta)$ can be written as follows :

(a) For $m=n, n+1, \cdots$,

$$(n-1)!\mu_{n,\delta}^{(m)}=m!\sum_{h=0}^{n-1}\sum_{j=0}^{h}\sum_{r=0}^{j}\binom{n-1}{r}\binom{n-1-j}{h-j}\frac{(n-1-j)!}{(m-h)!}(-1)^{j}$$
$$S(0, n-1-r, n-1-j)\delta^{r+1}(\delta-1)\cdots(\delta-m+h+1)$$

(b) For $m=1, 2, \cdots, n-1$,

$$(n-1)!\mu_{n,\delta}^{(m)}=m!\sum_{h=0}^{n-1}\sum_{j=0}^{h}\sum_{r=0}^{j}\binom{n-1}{r}\binom{n-1-j}{h-j}\frac{(n-1-j)!}{(m-h)!}(-1)^{j}$$
$$S(0, n-1-r, n-1-j)\delta^{r+1}(\delta-1)\cdots(\delta-m+h+1)$$
$$+\sum_{k=0}^{m}\sum_{j=k}^{m}\binom{n-1}{k}\binom{n-1-j}{m-j}(n-1-j)!(-1)^{j}$$
$$S(0, n-1-k, n-1-j)\delta^{k}]$$

Proof. Note that, for fixed n & j,

$$t^{j}(1+t)^{n-1-j}(1+t)^{\delta}=t^{j}\sum_{p=0}^{n-1-j}\binom{n-1-j}{p}t^{p}\left[1+\sum_{q=1}^{\infty}\frac{\delta(\delta-1)\cdots(\delta-p+1)}{q!}t^{q}\right]$$
$$=\sum_{m=j}^{n-1}\binom{n-1-j}{m-j}t^{m}+\sum_{h=j}^{n-1}\sum_{m=h+1}^{\infty}\binom{n-1-j}{h-j}$$
$$\frac{\delta(\delta-1)\cdots(\delta-m+h+1)}{(m-h)!}t^{m}$$

Hence, it follows from Lemma 3. 1 that

$$(n-1)!\phi_{n}(t\,;\,\delta)=\sum_{m=0}^{n-1}\sum_{k=0}^{m}\sum_{j=k}^{m}\binom{n-1}{k}\binom{n-1-j}{m-j}(n-1-j)!(-1)^{j}$$
$$S(0, n-1-k, n-1-j)\delta^{k}t^{m}$$
$$+\sum_{m=1}^{\infty}\sum_{h=0}^{m\wedge(n-1)}\sum_{j=0}^{h}\sum_{r=0}^{j}\binom{n-1}{r}\binom{n-1-j}{h-j}$$
$$\frac{(n-1-j)!}{(m-h)!}(-1)^{j}S(0, n-1-r, n-1-j)\delta^{r+1}$$
$$(\delta-1)\cdots(\delta-m+h+1)t^{m},$$

where $m\wedge n=\min(m, n)$.

Using Lemma 3. 2, the m-th factorial moment $\mu_{n,\delta}^{(m)}$ can be computed at least for small m Detailed computations show that, for $n>2$,

$$(n-1)!\ \mu_{n,\delta}^{(1)}=(n-2)![(n-1)^{2}-S(0, n-1, n-2)]$$

$$\frac{(n-1)!}{2}\mu_{n,\delta}^{(2)}=\{(n-1)(n-3)!\ S(0, n-2, n-3)$$
$$-(n-2)!\ S(0, n-1, n-2)+\frac{(n-1)!}{2}\}\delta$$
$$+\binom{n-1}{2}(n-1)!-(n-2)(n-2)!S(0, \mathrm{n}-1, n-2)$$

$$+(n-3)!S(0,n-1,n-3).$$

Since it can be easily shown that, for any positive integer,

$$S(0,n+1,n)=\sum_{i=1}^{n} i,$$

$$S(0,n+2,n)=\frac{1}{2}\left(\sum_{i=1}^{n} i^2+\sum_{i=1}^{n} i^3\right),$$

the next theorem follows.

Theorem 3.1. The mean μ and the variance σ^2 of the lattice distribution on the set $L(n,\delta)$ are given by

$$\mu=\mu_{n,\delta}^{(1)}=n/2,$$

$$\sigma^2=\mu_{n,\delta}^{(2)}-\mu^2$$

$$=n/12,$$

for any $n>2$.

We note that the mean and the variance of the lattice distribution on the set $L(n,\delta)$ is independent of δ for $n\geq 3$.

Theorem 3.2. The m-th factorial moment $\mu_{n,\delta}^{(m)}$ of the lattice distribution on the set $L(n,\delta)$ does not depend on δ for $m<n$.

Proof. Let $c_1,\cdots,c_m$ denote the constants determined by

$$x(x-1)\cdots(x-m+1)=c_1x+c_2x^2+\cdots+c_mx^m$$

for all x. Then, the m-th factorial moment $\mu_{n,\delta}^{(m)}$ is given by

$$\mu_{n,\delta}^{(m)}=\frac{1}{(n-1!)}\sum_{j=0}^{n-1}\sum_{k=1}^{m} c_k\ (\delta+j)^k\sum_{i=0}^{j}(-1)^i\binom{n}{i}(\delta+j-1)^{n-1}$$

$$=\frac{1}{(n-1)!}\sum_{k=1}^{m} c_k\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}\sum_{j=1}^{n-1}(\delta+j)^k(\delta+n-i)^{n-1}.$$

It can be observed from Lemma 3.2 that $\mu_{n,\delta}^{(m)}$ is a polynomial in δ with degrees at most m. Now, for fixed n and $m\leq n-1$, let $H(\delta)$ be the polynomial defined for all real numbre δ such that

$$H(\delta)=\sum_{k=1}^{m} c_k\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}\sum_{j=i}^{n-1}(\delta+j)^k(\delta+j-1)^{n-1}$$

Then, we have

$$H(\delta+1)-H(\delta)=\sum_{k=1}^{m} c_k\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}\Big[\sum_{j=i+1}^{n}(\delta+j)^k(\delta+j-i)^{n-1}$$

$$-\sum_{j=i}^{n-1}(\delta+j)^k(\delta+j-i)^{n-1}\Big]$$

$$=\sum_{k=1}^{m} c_k\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}[(\delta+n)^k(\delta+n-i)^{n-1}-(\delta+i)^k\delta^{n-1}]$$

$$=\sum_{k=1}^{m} c_k[(\delta+n)^k\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(\delta+n-i)^{n-1}$$

$$-\delta^{n-1}\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(\delta+i)^k].$$

By Lemma 2.1,

$$\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(\delta+n-1)^{n-1}=(-1)^{n-1}\delta^{n-1}.$$

Thus,

$$\begin{aligned}H(\delta+1)-H(\delta)&=-\delta^{n+1}\sum_{k=1}^{m}c_k\sum_{i=0}^{n}(-1)^i\binom{n}{i}(\delta+i)^k\\&=(-\delta)^{n-1}\sum_{k=1}^{m}c_kS(\delta,k,n)\\&=0\end{aligned}$$

Therefore,

$$H(l)-H(0)=0 \quad \text{for } l=0,1,\cdots,n,$$

which implies $H(\delta)-H(0)$, as a polynomial with degrees less than n, should be identically zero. Hence,

$$\mu_{n,\delta}^{(m)}=\mu_{n,0}^{(m)}$$

for all $\delta : 0\leq\delta\leq1$, and for $m<n$.

It should be noted that the m-th moment $\mu_{n,\delta}^{(m)}$ is independent of δ for $m<n$, and should be identical with the m-th moment of the random vairable S_n. The m-th moments $\mu_{n,\delta}^{(m)}$ for $n\leq4$ are tabulated in thable 1.

Table 1. Values of $\mu_{n,\delta}^{(m)}$

n \ m	0	1	2	3	4	…
1	1	e_1^*	e_1^*	e_1^*	e_1^*	…
2	1	1	e_2^*	e_2^*	e_2^*	…
3	1	3/2!	5/2!	e_3^*	e_3^*	…
4	1	2	13/3!	60/3!	e_4^*	…

$e_1^*=\delta^m$

$e_2^*=\delta^{m+1}+(\delta+1)^m(1-\delta)$

$e_3^*=[\delta^{m+2}+(\delta+1)^m(-2\delta^2+2\delta+1)+(\delta+2)^m(\delta^2-2\delta+1)]/2!$

$e_4^*=[\delta^{m+3}+(\delta+1)^m(-3\delta^3+3\delta^2+3\delta+1)+(\delta+2)^m(3\delta^3-6\delta^2+4)+(\delta+3)^m(-\delta^3+3\delta^2-3\delta+1)]/3!$

Remark : The results in this paper suggest a further study on the lattice distribution on $L(n,\delta)$. The natural questions are as follows; (a) What would be the limiting distribution of the lattice distribution as n gets large? (b) What distribution other than the uniform distribution can be reduced to the same distribution after operating convolutions and reducing modulo 1?

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