(Special Invited Paper)

On Minimal Sufficient Statistics†

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1. Introduction

Let (X,A) be a measurable space, i.e. X is a non-empty set and $\mathscr{A}$ is a σ-field of subsets of X. Let Ω be a parameter space and P be a family of probability measures P_θ, $\theta\in\Omega$ defined on $(X,\mathscr{A})$.

Let $\mathscr{A}_0$ be a sub-σ-field of $\mathscr{A}$, and let $\theta\in\Omega$ and $A\in\mathscr{A}$ be fixed. Then we can define the conditional probabiltiy $P_\theta\ (A|\mathscr{A}_0,x)$ of A given $\mathscr{A}_0$ as an $\mathscr{A}_0$-measurable function defined on X which satisfies

$$P_\theta(A\cap A_0)=\int_{A_0} P_\theta(A|\mathscr{A}_0,x)P_\theta(dx)$$

for any $A_0.\in\mathscr{A}_0$ $P_\theta(A|\mathscr{A}_0,.)$ is defined uniquely up to $(\mathscr{A}_0,P_\theta)$-equivalence.

If we can define $P_\theta(A|\mathscr{A}_0,.)$ for every $A\in\mathscr{A}$ independent of $\theta\in\Omega$, then we denote it by $P_\Omega(A|\mathscr{A}_0,.)$ and in this case $\mathscr{A}_0$ is called a sufficient σ-field of $\mathscr{A}$.

Next we shall define a necessary σ-field. For this purpose we first introduce the definition of the contraction. A set $N\in\mathscr{A}$ is termed an $(\mathscr{A},P)$-null set, if $P_\theta(N)=0$ holds for all $\theta\in\Omega$. Let $\mathscr{A}_0$ and $\mathscr{A}_1$ be two sub-σ-fields of $\mathscr{A}$. If for every $A_0\in\mathscr{A}_1$ there exists an $A_1\in\mathscr{A}_1$ such that the symmetric difference

$$A_0\Delta A_1=(A_0-A_1)\cup(A_1-A_0)$$

is an $(\mathscr{A},P)$-null set, then we call $\mathscr{A}_0$ a contraction of $\mathscr{A}_1$ and we denote it by $\mathscr{A}_0\subset\mathscr{A}_1(\mathscr{A},P)$.

If a sub-σ-field $\mathscr{A}_0$ of $\mathscr{A}$ is a contraction of every sufficient σ-field of $\mathscr{A}$, then we call $\mathscr{A}_0$ a necessary σ-field of $\mathscr{A}$. Ifa sufficient σ-field $\mathscr{A}_0$ is also necessary, we call $\mathscr{A}_0$ a necessary and sufficient σ-field or a minimal sufficient σ-field. These notions were introduced by Bahadur (1954) who established the existence of a minimal sufficient σ-field under the assumption that P is dominated by a σ-finite measure.

Let Y be another non-empty set with or without its σ-field given in advance. If Y is not endowed with its σ-field in advance, then following Lehmann and Scheffé (1950) or Bahadur (1954) any mapping t from X into Y is called a statistic. A statistic of this type is called a statistic of the first kind or an S_1-statistic from $(X,\mathscr{A})$ into Y. Throughout the paper we define in this case the σ-field $\mathscr{B}^*$ of Y as

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$$\mathscr{B}^* = \{B : B \subset Y,\ t^{-1}B \in \mathscr{A}\} \tag{1}$$

where t^{-1} denotes the inverse image. If furthermore the induced σ-field

$$t^{-1}\mathscr{B}^* = \{t^{-1}B : B \in \mathscr{B}^*\}$$

is a sufficient σ-field of A, then we call t an S_1-sufficient statistic from $(X, \mathscr{A})$ into Y.

On the other hand if Y is endowed with its σ-field $\mathscr{B}$ given in advance, then following Halmos and Savage (1949) any $\mathscr{A} \to \mathscr{B}$ measurable mapping t from X into Y is called a statistic. A statistic of this type is called a statistic of the second kind or an S_2-statistic from $(X, \mathscr{A})$ into $(Y, \mathscr{B})$. If t is an S_2-statistic from $(X, \mathscr{A})$ into $(Y, \mathscr{B})$ and if the induced σ-field

$$t^{-1}\mathscr{B} = \{t^{-1}B : B \in \mathscr{B}\}$$

is a sufficient σ-field of $\mathscr{A}$, then we call t an S_2-sufficient statistic from $(X, \mathscr{A})$ into $(Y, \mathscr{B})$.

Clearly, if t is an S_1-statistic from $(X, \mathscr{A})$ into Y, then t is also an S_2-statistic from $(X, \mathscr{A})$ into $(Y, \mathscr{B}^*)$. If furthermore t is an S_1-sufficient statistic from $(X, \mathscr{A})$ into Y, then t is also an S_2-sufficient statistic from $(X, \mathscr{A})$ into $(Y, \mathscr{B}^*)$.

Let t be an S_2-sufficient statistic from $(X, \mathscr{A})$ into $(Y, \mathscr{B})$. According to Lemma 2 of Halmos and Savage (1949), the conditional probability $P_\Omega(A/t^{-1}\mathscr{B}, x)$ of A given $t^{-1}\mathscr{B}$ can be identfied as $P_\Omega(A/t(x))$, where $P_\Omega(A/y)$ is a $\mathscr{B}$-measurable function defined on Y which statisfies

$$P_\theta(A \cap t^{-1}B) = \int_B P_\Omega(A/y) P_\theta(t^{-1}(dy))$$

for all $\theta \in \Omega$ and $B \in \mathscr{B}$, that is, $P_\Omega(A/y)$ is a conditional probability of A given the sufficient statistic t which is defined independent of $\theta \in \Omega$.

Let Y and Z be two-empty sets with or without their σ-fields $\mathscr{B}$ and $\mathscr{C}$given in advance, and let t and u be both S_1-or both S_2-statistics from $(X, \mathscr{A})$ into Y and Z respectively. If there exist an $(\mathscr{A}, P)$-null set N and a mapping f from Z into Y such that

$$t(x) = f(u(x)) \text{ for all } x \in X - N, \tag{2}$$

then we call t a functional contraction of the first kind or an F_1-contraction of u.

If furthermore f in (2) is $\mathscr{C} \to \mathscr{B}$ measurable, then we call t a functional contraction of the second kind or an F_2-contraction of u, with the understanding mentioned above that when $\mathscr{B}$ and $\mathscr{C}$ are not given in advance, $\mathscr{B}$ is put to $\mathscr{B}^*$ in (1) and $\mathscr{C}$ is similarl yput to

$$\mathscr{C}^* = \{C : C \subset Z,\ u^{-1}C \in \mathscr{A}\}.$$

Needless to say, if t is an F_2-contraction of u, then t is also an F_1-contraction of u.

Let t be an S_i-statistic ($i=1$ or 2). If t *is* an F_j-contraction ($j=1$ or 2) of every S_i-sufficient statistic, then we call t an (S_i, F_j)-necessary statistic. If an S_i-sufficient statistic t is (S_i, F_j)-necessary, then we call t an (S_i, F_j)-necessary and sufficient statistic or an (S_i, F_j)-minimal sufficient statistic.

Lehmann and Scheffé (1950) established the existence of an (S_1, F_1)-minimal sufficient statistic under the assumptiion that Ω is separable under the pseudo-metric

$$d(\theta_1, \theta_2)=\sup^{A\in\mathscr{A}}|P_{\theta_1}(A)-P_{\theta_2}(A)|. \qquad (3)$$

Further results on minimal sufficien statistics are also obtained in terms of S_1-statistics and F_1-contractions, among others:

(a) Bahadur and Lehmann (1955) give an example of $(X, \mathscr{A}, P)$ for which there exist both minimal sufficient σ-fields and (S_1, F_1)-minimal sufficient statistics, but $\mathscr{A}_0 \neq t^{-1}\mathscr{B}^*(\mathscr{A}, P)$ for any minimal sufficient σ-field $\mathscr{A}_0$ and any (S_1, F_1)-minimal sufficienl statistic t.

(b) Bahadur (1955) gives an example of $(X, \mathscr{A}, P)$ for which there exists a minimal sufficient σ-field $t^{-1}\mathscr{B}^*$, where t is an S_1-statistic but not (S_1, F_1)-minimal sufficient.

(c) Pitcher (1957) gives an example of $(X, \mathscr{A}, P)$ for which there does not exist a minimal sufficient σ-field nor an (S_1, F_1)-minimal sufficient statistic.

(d) Landers and Rogge (1972) give two examples of $(X, \mathscr{A}, P)$, one of which has a minimal sufficient σ-field but not an (S_1, F_1)-minimal sufficient statistic, the other has an (S_1, F_1)-minimal sufficient but not a minimal sufficient σ-field.

In Section 2 we shall give some lemmas. In Section 3 we shall show that in some cases different conclusions may be obtained by using different definitions of statistics S_1- or S_2- and of functional contractions F_1- or F_2-, concerning the relations between statistics and sub-σ-fields and especially between minimal sufficient statistics and minimal sufficient σ-fields.

2. Lemmas

We call $(\mathscr{A}, P)$ completed if any subset of an arbitrary $(\mathscr{A}, P)$-null set belongs to (hence (A, P)-null).

Lemma 1. Let $(\mathscr{A}, P)$ be completed. Ift and u are S_1-statistics from $(X, \mathscr{A})$ into Y and Z respectively, and if t is an F_1-contraction of u, then t is also an F_2-contraction of u, putting $\mathscr{B}=\mathscr{B}^*$ and $\mathscr{C}=\mathscr{C}^*$.

Proof. By the assumption there exist an $(\mathscr{A}, P)$-null set N and a mapping f from Z into Y such that (2) holds. We shall show that f is $\mathscr{C}^*\to\mathscr{B}^*$ measurable.

Let B be an arbitrary element of $\mathscr{B}^*$, then we have from (2) that

$$t^{-1}B\Delta u^{-1}(f^{-1}B)\subset N. \qquad (4)$$

Since $t^{-1}B\in\mathscr{A}$ by the definition of $\mathscr{B}^*$, the assumption of the completedness of $(\mathscr{A}, P)$ and the inclusion relation (4) imply that $u^{-1}(f^{-1}B)\in\mathscr{A}$, which means $f^{-1}B\in\mathscr{C}^*$ by the definition of $\mathscr{C}^*$. Thus the $\mathscr{C}^*\to\mathscr{B}^*$ measurability of f is proved.

Lemma 2. Let $(\mathscr{A}, P)$ be completed. If t is an (S_1, F_1)-minimal sufficient statistic from $(X,\mathscr{A})$ into Y, then t is also an (S_1,F_2)-minimal sufficient statistic from $(X,\mathscr{A})$ into $(Y,\mathscr{B}^*)$.

Proof. Use Lemma 1 to prove the (S_1,F_2)-necessity of t.

Lemma 3. Let P be a dominated family If t is an S_2-sufficient statistic from $(X,\mathscr{A})$

into $(Y,\mathscr{B})$, then t is als an S_1-sufficient statistic from $(X,\mathscr{A})$ into Y.

Proof. By the assumption $t^{-1}\mathscr{B}$ is a sufficient σ-field of $\mathscr{A}$, hence $t^{-1}\mathscr{B}\subset t^{-1}\mathscr{B}^*$ implies the sufficiency of $t^{-1}\mathscr{B}^*$ by Theorem 6. 4 of Bahadur (1954).

Lemma 4. Let P be a dominated family. If t is an (S_1,F_1)-minimal sufficient statistic from $(X,\mathscr{A})$ into Y, then t is also an (S_2,F_1)-minimal sufficient statistic from $(X,\mathscr{A})$ into $(Y,\mathscr{B}^*)$.

Proof. S_2-sufficiency of t is evident. Let u be an arbitrary S_2-sufficient statistic from $(X,\mathscr{A})$ into $(Z,\mathscr{C})$. Then by Lemma 3 u is an S_1-sufficient statistic from $(X,\mathscr{A})$ to Z, hence by the assumption of (S_1,F_1)-necessity of t, t is an F_1-contraction of u, whith completes the proof.

Lemma 5. Let P be a dominated family. If t is an (S_2,F_1)-minial sufficient statistic from $(X,\mathscr{A})$ into $(Y, \mathscr{B})$, then t is also an (S_1, F_1)-minimal sufficient statistic from $(X,\mathscr{A})$ into Y.

Proof. S_1-sufficiency of t follows from Lemma 3. Let u be an arbitrary S_1-sufficient statistic from $(X,\mathscr{A})$ into Z, then u is also an S_2-sufficient statistic from $(X,\mathscr{A})$ into $(Z,\mathscr{C}^*)$, hence by the assumption of (S_2,F_1)-necessity of t, t is an F_1-contraction of u, which completes the proof.

3. Main results

In this section we pose eight problems concerning statistics, sub-σ-fields, minimal sufficient statistics and minimal sufficient σ-fields, and give them answers as far as possiqle. Answers are expressed by Yes or No, dedending on the definitions of statistics S_1-or S_2-and of functional contractions F_1-or F_2-given in parentheses.

Problem 1. Does the functional contraction of statistics imply the contraction of the σ-fields induced by them?

No (S_1,F_1)——Example 1 of Bahadur (1955) is a counterexample to the problem.

No (S_2,F_1)——The above example of Bahadur can be used as a counterexample also in this case, putting $\mathscr{B}=\mathscr{B}^*$ and $\mathscr{C}=\mathscr{C}^*$.

Yes (S_2,F_2)——Let f be $\mathscr{C}\to\mathscr{B}$ measurable mapping from Z into Y and let (2) hold for some $(\mathscr{A},P)$-null set N. Let $t^{-1}B$, $B\in\mathscr{B}$ be an arbitrary element of $t^{-1}\mathscr{B}$, then we have $f^{-1}B\in\mathscr{C}$ because f is $\mathscr{C}\to\mathscr{B}$ measurable, and we get furthermore $u^{-1}\ (f^{-1}B)\in u^{-1}\mathscr{C}$ and (4). Hence $t^{-1}\mathscr{B}\subset u^{-1}\mathscr{C}\ (\mathscr{A},P)$ is proved.

Yes (S_1,F_2)——The answer follows from the answer in the case (S_2,F_2), putting $\mathscr{B}=\mathscr{B}^*$ and $\mathscr{C}=\mathscr{C}^*$.

Problem 2. Does the contraction of the σ-fields induced by statistics imply the functional contraction of the statistics?

No (S_1,F_1)——Example 2 of Bahadur (1955) is a counterexample to the problem.

No (S_1,F_2)——The above example of Bahadur can be used as a counterexample also in this case, for if t is not an F_1-contraction of u, then t is not an F_2-contraction of u

a fortiori.

No (S_2,F_1)——The same example of Bahadur is available as a counterexample also in this case, putting $\mathscr{B}=\mathscr{B}^*$ and $\mathscr{C}=\mathscr{C}^*$.

No (S_2,F_2)——The above counterexample in the case (S_1,F_1) is available *a fortiori* as a counterexample also in this case.

Problem 3. Does the minimal sufficiency of a statistic imply the minimal sufficiency of the σ-field induced by it?

No (S_1,F_1)——The example of Bahadur and Lehmann (1955) is a counterexample to the problem (See (a) in the Introduction).

No (S_1,F_2)——As $(\mathscr{A},P)$ in the above example of Bahadur and Lehmann is completed, the (S_1,F_1)-minimal sufficient statistic in the example is by Lemma 2 (S_1,F_2)-minimal sufficient. Hence the same example can be used a counterexample also in this case.

No (S_2,F_1)——Since the family P in the same example is dominated, the (S_1,F_1)-minimal sufficient statistic in the example is by Lemma 4 (S_2,F_1)-minimal sufficient, putting $\mathscr{B}=\mathscr{B}^*$. Hence the same example is available as a counterexample also in this case.

Yes (S_2,F_2)——Let t be an S_2-sufficient statistic, then the sufficiency of $t^{-1}\mathscr{B}$ follows from the definition. Let $\mathscr{A}_0$ be any sufficiet σ-field of $\mathscr{A}$. Let u be the statistic of the identity from $(X,\mathscr{A})$ onto $(Z,\mathscr{C})=(X,\mathscr{A}_0)$, then u is an S_2-sufficient statistic since $u^{-1}\mathscr{C}=\mathscr{A}_0$ is a sufficient σ-field. As t is an (S_2,F_2)-necessary statistic, t is an F_2-contraction of u, hence by Yes-answer of Problem 1 in the case (S_2,F_2) $t^{-1}\mathscr{B}$ is a contraction of $u^{-1}\mathscr{C}=\mathscr{A}_0$. Thus the necessity of $t^{-1}\mathscr{B}$ is proved, which completes the proof.

Note. No-answers in the cases (S_1,F_1), (S_1,F_2) and (S_1,F_1) follow also from No-answers of Problem 5 given below.

Problem 4. Does the minimal sufficiency of the σ-field induced by a statistic imply the minimal sufficiency of the statistic?

No (S_1,F_1)——Example 2 of Bahadur (1955) is a counterexample to the problem.

No (S_1,F_2)——The above example of Bahadur can be used as a counterexample also in this case, because the statistic in the example is not (S_1,F_2)-necessary *a fortiori.*

No (S_2,F_1)——Two S_1-sufficient statistics t and u from $(X,\mathscr{A})$ into Y and into Z are also S_2-sufficient statistic from $(X,\mathscr{A})$ into $(Y,\mathscr{B}^*)$ and into $(Z,\mathscr{C}^*)$ respectively. Hence the same example is available as a counterexample also in this case.

No (S_2,F_2)——The above counterexample in the case (S_2,F_1) is available *a fortiori* as a counterexample also in this case.

Problem 5. Does the existence of a minimal sufficient statistic implythe existence of a minimal sufficient σ-field?

No (S_1,F_1)——Example 2 of Landers and Rogge (1972) is a counterexample to the problem.

No (S_1,F_2)——Modify the above example of Landers and Rogge as follows. In their notation modify $\mathscr{A}$ to the σ-field generated by the set T and the σ-field $\mathscr{A}^{(0)}$ of all Lebesgue measurable sets A for which

$$A_r=\{s\in[0,1]:(s,r)\in A\}$$

is Lebesgue measurable for all $r\in[0,1]$. By defining probability measures $Q_r, P_r, r\in[0,1]$ and P as in their paper using $\mathscr{A}^{(0)}$ instead of $\mathscr{B}\times\mathscr{B}$, we can make $(\mathscr{A},P)$ completed.

In proving their proposition (i) we have only to modify $\mathscr{B}\times\mathscr{B}$ to $\mathscr{A}^{(0)}$. The proof of their proposition (ii) requires no modifications. Thus we have again a counterexample in the case (S_1,F_1) with $(\mathscr{A},P)$ this time completed. Hence by Lemma 2 the modified counterexample is available as a counterexample also in this case.

No (S_2,F_1)——Modify slightly the proposition (ii) in the same example of Landers and Rogge to the following: the idetity I viewing as an S_2-statistic from $(X,\mathscr{A})$ onto $(X,\mathscr{A})$ is (S_2,F_1)-minimal sufficient.

To prove this modified proposition, let U be an S_2-sufficient statistic from $(X,\mathscr{A})$ into $(Z,\mathscr{C})$, then $\mathscr{B}\times\mathscr{B}$ in their notation is a contraction of $U^{-1}\mathscr{C}$, which can be seen by the same reasoning as the proof of their (a) since $U^{-1}\mathscr{C}$ is a sufficient σ-field. Furthermore we have

$$U^{-1}\mathscr{C}\subset U^{-1}\mathscr{C}^*=\mathscr{A}_U \text{ (in their notation)},$$

so $\mathscr{B}\times\mathscr{B}$ is again a contraction of $\mathscr{A}_U$. The rest of the proof requires no modification. Thus the same example is available as a counterexample also in this case.

Yes (S_2,F_2)——The answer follows from Yes-answer of Problem 3.

Problem 6. Does the existence of a minimal sufficient σ-field imply the existence of a mininal sufficient statistic?

No (S_1,F_1)——Example 1 of Landers and Rogge (1972) is a counterexample to the problem.

No (S_1,F_2)——The above example of Landers and Rogge can be used as a counterexample also in this case, because there does not exist an (S_1,F_2)-sufficient statistic in the example *a fortiori*.

No (S_2,F_1)——Since the family of probability measures in the same example of Landers and Rogge is dominated, there does not exist an (S_2,F_1)-minimal sufficient statistic because of Lemma 5. Therefore the same example is available as a counterexample also in this case.

No (S_2,F_2)——The above counterexample in the case (S_2,F_1) is available *a fortiori* asa counterexample also in this case.

Proble 7. Does there exist a minimal sufficient statistic when Ω is separable under the pseudo-metric (3)?

Yes (S_2,F_2)——Let $\{\theta_1,\theta_2,\cdots\}$ be a countable dense subset of Ω under the pseudo-metric (3), and let $\lambda=\sum_{i=1}^{\infty}2^{-i}P_{\theta_i}$ be a probability measure defined on (X,A) which is equivalent to P in the sense of Halmos and Savage (1949). Let a set of versions of Radon-Nikodym derivatives $dP_\theta/d\lambda=p\ (\cdot,\theta)$ be fixed. Let Y be the countable product space of the sets of real numbers

$$Y=R\times R\times\cdots$$

and let B the σ-field generated by all cylinder sets of the form

$$\underbrace{R}_{i}\times\cdots\times \underbrace{R}_{i-1}\times \underbrace{Bi}_{i}\times \underbrace{R}_{i+1}\times\cdots,\quad i=1,2,\cdots$$

where B_i is any Borel subset of R. Then

$$t(x)=(p(x,\theta_1),p(x,\theta_2),\cdots) \tag{5}$$

is shown to be an (S_2,F_2)-minimal sufficient statistic. The idea of the statistic (5) is due to Bahadur (1954). See also Nabeya (1978) for the detailed proof.

Yes (S_2,F_1)——The above t in (5) is (S_2,F_1)-minimal sufficient *a fortiori.*

Yes (S_1,F_1)——The original proof was given by Lehmann (1950). The same conclusion can also be deduced from the case (S_2,F_1). For, since Ω is separable under the pseudo-metric (3), P is dominated, hence by Lemma 5 the above t is (S_1,F_1)-minimal sufficient, because it was (S_2,F_1)-minimal sufficient.

Open (S_1,F_2)——S_1-sufficiency of the above t in (5) is clear from the case (S_1,F_1). Let u be an arbitrary S_1-sufficient statistic from $(X,\mathscr{A})$ into Z. As t is an (S_2,F_2)-necessary statistic from $(X,\mathscr{A})$ into $(Y,\mathscr{B})$ in the notation of the case (S_2,F_2), t is an (S_2,F_2)-contraction of u if the underlying σ-fields of Y and Z are $\mathscr{B}$ and $\mathscr{C}^*$ respectively, but I have not been able to prove that t is an (S_2,F_2)-contraction of u if the underlying σ-field of Y is changed to $\mathscr{B}^*$.

A partial answer in this case is Yes, if $(\mathscr{A},P)$ is completed, as is seen from Lemma 1.

Before giving the final problem we shall introduce the definition of completeness of sub-σ-fields and statistics.

Let $\mathscr{A}_0$ be a sub-σ-field of $\mathscr{A}$. If f is a real-valued $\mathscr{A}_0$-measurable function, and if

$$\int_X f(x)P_\theta(dx)=0 \text{ for all } \theta\in\Omega \tag{6}$$

implies

$$f(x)=0 \text{ a.e. } (\mathscr{A}_0,P_\theta) \text{ for all } \theta\in\Omega, \tag{7}$$

then we call $\mathscr{A}_0$ a complete σ-field. If f is a real-valued bounded $\mathscr{A}_0$-measurable function, and if (6) implies (7), then we call A_0 a boundedly complete σ-field. Needless to say, if $\mathscr{A}_0$ is complete then it is boundedly complete.

Let t be an S_1-or S_2-statistic, and if t is an S_1-statistic, then we put $\mathscr{B}=\mathscr{B}^*$ as usual. If the induced σ-field $t^{-1}\mathscr{B}$ is complete or boundedly complete, then t is called a complete or boundedly complete statistic respectively.

Bahadur (1957) shows that if $\mathscr{A}_0$ is a sufficient and boundedly complete σ-field, then $\mathscr{A}_0$ is minimal sufficient. The last problem roughly corresponds to this theorem of Bahadur.

Problem 8. Do the sufficiency and the completeness of a statistic imply the minimal sufficiency of the statistic?

No (S_1)——The statistic f in Example 2 of Bahadur (1955) is sufficient and complete, but not minimal sufficient.

No (S_2)——The same example is available as a counterexample also in this case, putting $\mathscr{B}=\mathscr{B}^*$ in my notation.

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