General Properties of Quantization Systems with a Stochastic Reference

(Stochastic Reference를 가진 量子化 시스템의 一般的인 성질)

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要　約

Stochastic reference를 가진 두개의 量子化 시스템을 분석하고 비교하였다. 量子化된 出力信号에 対하여 invariance 성격을 갖기 위한 조건이 서로 같다는 것이 보여졌다. Stochastic reference 신호를 이용한 polarity coincidence 방법에 의한 상관관계 함수 추정이 여기서 구한 一般的 성질의 특수한 경우이다. 과거의 stochastic computing은 여기서 고려한 첫 번째 system으로부터 나오고 그리고 L.G Robert가 이용한 특성은 두 번째 시스템의 일반적 성질의 특수한 경우를 이용했다는 것을 보였다.

Abstract

This paper deals with two quantization systems with a stochastic refernce and gives the unified statistical properties of the two systems. The conditions are derive for the invariance of the output quantized signal with respect to the input signal for the two systems and it is shown that they are same.

The correlation function estimation by a polarity method using stochastic reference signals is shown to be a special case of the general properties derived here.

We have also shown that the classical stochastic computing is derived from the general properties of the first system and that L.G. Roberts has used a special characteristic of the general properties of the second system in his image processing.

I. Introduction

Generally the quantization system is used for the digital signal processing. The quantization system used most frequently is a uniform quantizer. As far as concerned with a quantizer, one cannot avoid a quantization noise. Generally the quantization noise has a bad effect upon the quality of image or speech because of its dependence on the input signal of quantizer.[1][22][27]

A special case of a general quantizer, one bit quantizer, has been used for a long time for the correlation function estimation.[3][4] Generally it is very difficult to obtain the correlation function of an arbitrary signal. But the correlation function of the output of an one-bit quantizer can be found easily. Because the output of an one bit quantizer is consisted of the 0 or 1 in the logical sense. It is necessary to have only some shift registers, some AND gates and some counters for the correlation function of the output. For the correlation function of the input, one must use a inversion formula between the cor-

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relation functions of the input and the output.[3][31]

Especially one bit quantizer with a stochastic reference was used for the estimation of correlation function because of the unbiased output.[5][6][7] In this case, the output correlation function is the same as the input correlation function. Therefore it is not necessary to use a inversion formula.

KE-YEN CHANG et al, have considered one bit quantizer with a stochastic reference for the correlation function under the name of the modified digital correlator. They proved that the correlation function estimation of the quantized signals is equal to the true correlation function for arbitrary inputs under the use of a stochastic reference signal with some special conditions. CASTANIE F. has realized a correlator using a stochastic reference and considered the performance.[11][12] He also used a stochastic reference for the spectral analysis.[13][14] In the other part, S. T. RIBEIRO has proposed a new type of the computing machines using stochastically coded signals.[10] In his paper, he has used the fact that the continuous variable is proportional to a probability of a pulse occurrence at a certain sampling time when the continuous variable is coded stochastically using one bit quantizer with a stochastic reference. After him, much has been studied in this domain.[15]-[20]

In 1962, L. G. Roberts has considered a uniform quantizer with a stochastic reference (so-called "dither"), which has been subtracted from the quantized signal at the output (we call it system 2). He has used this system for picture coding.[1]

L. SCHUCHMAN has derived the conditions that a dither signal must meet so that the quantizer noise can be considered independent of the signal.[2] After L.G. Roberts, many people have studied the effect of dither on the quantized visual signals and on the quantized speech signals.[21]-[28]

S.S. AN has derived the second order characteristic of dither which gives the optimum signal to noise ratio after a lowpass filter connected to the output of L.G. Robert's system (System 2 of Fig. 2).[30]

But in spite of the similarity of the two systems, they have not been analyzed theoretically in the unified views.

In this paper we analyzed the two systems and derived general properties from which it is shown that the conditions for the invariance of the output quantized signal with respect to the input signal for the two systems are same. ..d we have shown that the correlation function estimation by a polarity method using a stochastic reference signal is a special case of the general properties derived here. It is also shown that the classical stochastic computing is derived from the general properties of System 1 and that L.G. Roberts has used a special characteristic of the general properties of System 2 in his image processing. (See Sec II and III)

In Sec IV, we discuss the general properties derived and suggest further applications.

The proofs of the properties are given in Appendix.

II. Statistical properties of quantizer with a stochastic reference

The two systems we have investigated are shown in Fig. 2.

System 1 is that used in St. Ribero's paper [10] in the case of one bit quantizer. There, he has applied that system to make a stochastically coded signal for an arbitrary input signal, and he has used that signal to do addition, multiplication, etc. (This computing method is called a stochastic computing.)

This will be restated in Sec. III.

System I has been also used in image processing.[22,23] System 2 is often used in low bit rate PCM or in image processing because of its smoothing effect with respect to quantization noise.

Up to now, in spite of many people's research, the properties for the two systems were stated dispersedly and only partly. In some papers[1,2], the first order characteristics were only considered. From these one can not derive the second order characteristics from which comes the concept of frequency. In some others,[27,28] only the qualitative effects of dither were investigated. Here we have presented for the two systems the unified statistical properties which contain some new properties and some known properties. We think that all the statistical properties can be derived from the equations presented here as far as concerned with the two systems. (The applications of the equations are considered in Sec III and Sec IV.)

The quantizer characteristic is given in Fig. 1, which shows a uniform quantizer. (One can show that the benefits with a stochastic reference come from both a uniform quantization characteristic and a characteristic of stochastic reference.)

The main equations which give general properties are conditional characteristic functions of the output of the quantizer and of the quantization noise, and conditional moments of those. In Table 1, we have summarized conditional characteristic functions of Y(t), n(t) and y(t) for given x(t), conditional moment, etc. where n(t) is the quantization noise. (We call these properties first order properties.) In Table 2, joint conditional characteristic functions are summarized in addition to conditional moments, etc. (These will be called second order properties.)

The proofs of these equations are in Appendix.

To facilitate the comprehension of the tables, we give definitions and short explanations of the notations used.

Definitions and Short explanations

x(t) ; Input signal of the quantizer (See Fig. 2)

d(t) ; Stochastic reference (See Fig. 2)

n(t) ; Quantization noise (See Fig. 2)

Y(t), y(t) ; Output signals for each system (See Fig. 2)

$\Gamma_Y(\omega|x) \triangleq E(e^{j\omega Y}|x)$; Conditional characteristic fcn. of Y for given x.

$\Gamma d\,(\omega) \triangleq E(e^{j\omega d})$; Characteristic fcn of d(t)

$\Gamma n\,(\omega|x) \triangleq E(e^{j\omega n}|x)$; Conditional characteristic fcn. of n for given x.

E(Y|x) ; Conditional expectation of Y for given x.

LS(m) ; $\Gamma d^{(q)}\,(\frac{\pi k}{\delta}) = 0$ for $q = 0, 1, \cdots, m$ and $k \neq 0$ where $\Gamma d^{(q)}\,(\frac{\pi k}{\delta})$ means qth order derivative of $\Gamma d(\omega)$ at $\omega = \frac{\pi k}{\delta}$. This condition constrains the statistical properties which a stochastic reference can have.

$\Gamma Y(\omega_1, \omega_2|x_t, x_s) \triangleq E(e^{j\omega_1 Y_t + j\omega_2 Y_s}|x_t,x_s)$; Joint conditional characteristic fcn. of Y(t), Y(s) for given x(t), x(s).

$\Gamma d(\omega_1, \omega_2) \triangleq E(e^{j\omega_1 d_t + j\omega_2 d_s})$; Joint characteristic fcn. of d(t), d(s).

$\Gamma n\,(\omega_1,\omega_2|x_t,x_s) \triangleq E(e^{j\omega_1 n_t + j\omega_2 n_s}|x_t,x_s)$; Joint conditional characteristic fcn. of n(t), n(s) for given x(t), x(s)

LS (m,n) ; $\Gamma d^{(p)(q)}\,(\frac{\pi k}{\delta}, \frac{\pi l}{\delta}) = 0$ for $p = 0, 1, \cdots, m$ and $q = 0, 1, \cdots, n$ and $k, l \neq 0$.

This condition constrains the statistical properties which a stochastic reference can have.

$\mathrm{sinc}\,(x) \triangleq \sin \pi x/\pi x$

$\omega \triangleq 2\pi f$; In Table 1 and 2 we have used both ω and f for the simplicity of notations.

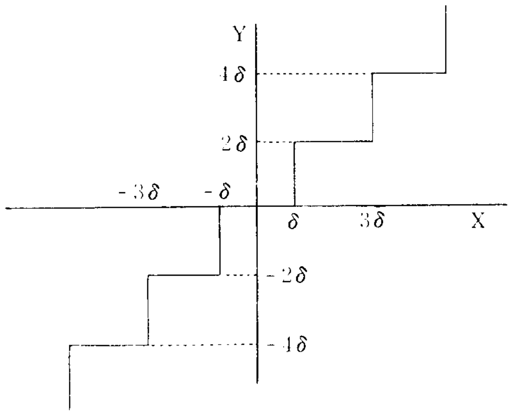
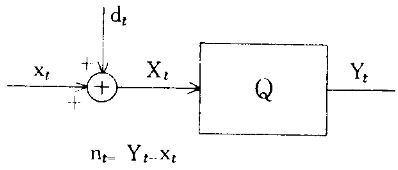
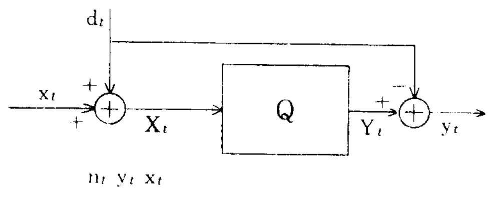


Fig. 1. Characteristic function of the quantizer, Q.



System 1



System 2

Fig. 2. Two systems investigated.

Table 1. First order properties.

System 1	System 2
Conditional characteristic fcns. of Y(t), n(t) for given x(t)	Conditional characteristic fcns. of y(t), n(t) for given x(t)
(1) $\bar{\Gamma}_Y(\omega\|x_t) = \sum_k e^{-jx_t(\omega - \frac{\pi k}{\delta})} \Gamma_d(-\omega+\frac{\pi k}{\delta}) \mathrm{sinc} 2\delta(f-\frac{k}{2\delta})$	(1)′ $\bar{\Gamma}_y(\omega\|x_t) = \sum_k e^{-jx_t(\omega - \frac{\pi k}{\delta})} \Gamma_d(\frac{\pi k}{\delta}) \mathrm{sinc} 2\delta(f-\frac{k}{2\delta})$
(2) $\bar{\Gamma}_n(\omega\|x_t) = \sum_k e^{j\frac{\pi k}{\delta}x_t} \Gamma_d(-\omega+\frac{\pi k}{\delta}) \mathrm{sinc}\, 2\delta(f-\frac{k}{2\delta})$	(2)′ $\bar{\Gamma}_n(\omega\|x_t) = \sum_k e^{j\frac{\pi k}{\delta}x_t} \Gamma_d(\frac{\pi k}{\delta}) \mathrm{sinc}\, 2\delta(f-\frac{k}{2\delta})$
Conditional expectation of moments of Y(t), n(t) under $d(t) \in LS(m)$	Conditional expectation of moments of y(t), n(t) under $d(t) \in LS(o)$
(3) $E(Y_t^m\|x_t) = \sum_{i=0}^{m}\sum_{q=0}^{m-i} \frac{m!}{i!q!(m-i-q)!} \frac{\delta^i}{i+1} E(d_t^q)\, x_t^{m-i-q}$ i : even	(3)′ $E(y_t^m\|x_t) = \sum_{i=0}^{m} {}_mC_i \frac{\delta^i}{i+1} x_t^{m-i}$ i : even
(4) $E(n_t^m\|x_t) = \sum_{i=0}^{m} {}_mC_i \frac{\delta^i}{i+1} E(d^{m-i})$ i : even	(4)′ $E(n_t^m\|x_t) = \frac{\delta^m}{m+1}$ m : even
Conditional expectation of first, second order moments	Conditional expectation of first, second order moments
(5) $E(Y_t\|x_t) = x_t + E(d_t)$ $E(Y_t^2\|x_t) = x_t^2 + 2E(d_t)x_t + E(d_t^2) + \frac{\delta^2}{3}$	(5)′ $E(y_t\|x_t) = x_t$ $E(y_t^2\|x_t) = x_t^2 + \frac{\delta^2}{3}$
(6) $E(n_t\|x_t) = E(dt)$ $E(n_t^2\|x_t) = E(d_t^2) + \frac{\delta^2}{3}$	(6)′ $E(n_t\|x_t) = 0$ $E(n_t^2\|x_t) = \frac{\delta^2}{3}$

Remark :

For the first order properties, the most general properties are Eqs. (1), (1)′, (2), (2)′,. Because Eqs. (3), (3)′, (4), (4)′ can be derived from them (see Appendix). As one can see, Eqs. (5), (5)′, (6), (6)′ are the special cases of Eqs. (3), (3)′, (4), (4)′. If we are concerned with the quantization noise power only, the most remarkable difference between the two systems can be found from Eqs. (6) and (6)′.

Under the condition of $d(t) \in LS(2)$, the quantization noise power of System 1 is always larger than that of System 2. (The condition of LS(2) implies LS(0) by the definition.)

But System 2 is more complex than System 1 because of the substraction of the stochastic reference at the output of the quantizer. Generally to circumvent that difficulty of System 2, one use a pseudo-random noise as a stochastic reference.[1]

Table 2. Second order properties.

Joint conditional characteristic fcns. of Y(t), Y(s), n(t), n(s)	Joint conditional characteristic fcns. of y(t), y(s), n(t), n(s)
(7) $\Gamma_Y(\omega_1, \omega_2 \mid x_t, x_s)$ $= \sum_{k,l} e^{-jx_t(\omega_1-\frac{\pi k}{\delta}) - jx_s(\omega_2-\frac{\pi l}{\delta})}$ $\Gamma_d(-\omega_1+\frac{\pi k}{\delta}, -\omega_2+\frac{\pi l}{\delta})\,\mathrm{sinc}\,2\delta(f_1-\frac{k}{2\delta})\,\mathrm{sinc}\,2\delta(f_2-\frac{l}{2\delta})$	(7)′ $\Gamma_y(\omega_1, \omega_2 \mid x_t, x_s)$ $= \sum_{k,l} e^{-jx_t(\omega_1-\frac{\pi k}{\delta}) - jx_s(\omega_2-\frac{\pi l}{\delta})}$ x $\Gamma_d(\frac{\pi k}{\delta}, \frac{\pi l}{\delta})\,\mathrm{sinc}\,2\delta(f_1-\frac{k}{2\delta})\,\mathrm{sinc}\,2\delta(f_2-\frac{l}{2\delta})$
(8) $\Gamma_n(\omega_1, \omega_2 \mid x_t, x_s)$ $= \sum_{k,l} e^{j\frac{\pi k}{\delta} x_t + j\frac{\pi l}{\delta} x_s}$ $\Gamma_d(-\omega_1+\frac{\pi k}{\delta}, -\omega_2+\frac{\pi k}{\delta})\,\mathrm{sinc}\,2\delta(f_1-\frac{k}{2\delta})\,\mathrm{sinc}\,2\delta(f_2-\frac{l}{2\delta})$	(8)′ $\Gamma_n(\omega_1, \omega_2 \mid x_t, x_s)$ $= \sum_{k,l} e^{j\frac{\pi k}{\delta} x_t + j\frac{\pi l}{\delta} x_s}$ $\Gamma_d(\frac{\pi k}{\delta}, \frac{\pi l}{\delta})\,\mathrm{sinc}\,2\delta(f_1-\frac{k}{2\delta})\,\mathrm{sinc}\,2\delta(f_2-\frac{l}{2\delta})$
Conditional expectation of moments under the condition $d(t) \in LS(m, n)$	Conditional expectation of moments under the condition $d(t) \in LS(0, 0)$
(9) $E(Y_t^m Y_s^n \mid x_t, x_s) = \sum_{i=0}^{m} \sum_{q=0}^{m-i} \sum_{h=0}^{n} \sum_{r=0}^{n-h}$ $\frac{m!n!}{i!q!(m-i-q)!h!r!(n-h-r)!} \frac{\delta^{i+h}}{(i+1)(h+1)}$ $E(d_t^q d_s^r)\, x_t^{m-i-q}\, x_s^{n-h-r}$ i, h : even	(9)′ $E(y_t^m y_t^n \mid x_t, x_s) = \sum_{i=0}^{m} \sum_{h=0}^{n}$ $\frac{m!n!}{i!(m-i)!h!(n-h)!} \frac{\delta^{i+h}}{(i+1)(h+1)}$ $x_t^{m-i}\, x_s^{n-h}$ i, h : even
(10) $E(n_t^m n_t^n \mid x_t, x_s) = \sum_{i=0}^{m} \sum_{h=0}^{n}$ $\frac{m!n!}{i!(m-i)!h!(n-h)!} \frac{\delta^{i+h}}{(i+1)(h+1)} E(d_t^{m-i} d_s^{n-h})$ i, h: even	(10)′ $E(n_t^n n_s^n \mid x_t, x_s)$ $= \frac{\delta^{m+n}}{(m+1)(n+1)}$ m, n : even
Expectation of first or second order moments	Expectation of first or second order moments
(11) $E(Y_t Y_s \mid x_t, x_s) = x_t x_s + E(d_s) x_t$ $+ E(d_t) x_s + E(d_t d_s)$ $E(Y_t^2 Y_s \mid x_t, x_s) = x_t^2 x_s + E(d_s) x_t^2 + 2E(d_t) x_t x_s$ $+ 2E(d_t d_s) x_t + \frac{\delta^2}{3} x_s + \frac{\delta^2}{3} E(d_s) + E(d_t^2) x_s$ $+ E(d_t^2 d_s)$	(11)′ $E(y_t y_s \mid x_t, x_s) = x_t x_s$ $E(y_t^2 y_s \mid x_t, x_s) = x_t^2 x_s + \frac{\delta^2}{3} x_s$
(12) $E(n_t n_s \mid x_t, x_s) = E(d_t d_s)$ $E(n_t^2 n_s \mid x_t, x_s) = E(d_t^2 d_s) + \frac{\delta^2}{3} E(d_s)$	(12)′ $E(n_t n_s) = 0$, $E(n_t^2 n_s) = 0$ $E(n_t^2 n_s^2) = \frac{\delta^4}{9}$

One can see that the second order properties are the generalization of the first order properties. But the significance of each order properties is totally different. Because the concept of correlation or frequency does not come from the first order properties. (Of course, to give a frequency (or spectrum) concept, one must have a condition of stationarity in addition to the second order properties given here. The properties of (1), (1)', (2),(2)', (7), (7)',(8), (8)', were derived without any constraint. S.S. An[30] has further considered System 2 under the condition that the input is a first order Markov and stationary process.)

III. Application of the derived equations

As some applications of results of previous section, we consider three cases. They are the stochastic computing, the digital correlator and the dither in speech or image coding.

[1] Stochastic Computing

The stochastic computing is the method of computing (for example, addition, multiplication, etc.) using one bit quantizer with a stochastic reference.[10-20] (With a stochastic computer, one can do addition, multiplication, integration like an analog computer using digital circuits.)

From Eq. (5) of Table 1, we see

$$E(Y_t|x_t) = x_t + E(d_t) \tag{13}$$

under the condition of

$$\Gamma d\left(\frac{\pi k}{\delta}\right) = 0 \quad \text{for } k \neq 0 \tag{14}$$

This is the essential formula for the stochastic computing. Eq. (13) was derived from the multilevel uniform quantizer with a stochastic reference. Therefore it is also well suited for a one bit quantizer with a stochastic reference if the input signal x(t) is restricted to $[-\delta, \delta]$ and the stochastic reference d(t) satisfies the conditions (14) and $d(t) \in [-\delta, \delta]$.

One of the stochastic references satisfying these conditions is the uniformly distributed one.

In this case, Eq. (13) becomes

$$E(Y_t|x_t) = x_t \tag{15}$$

Eq. (15) shows that the conditional expectation of the output of one bit quantizer is the input signal. The stochastic computing considered by S.T. Ribeiro used this fact.[10]

In the other part, from Eq. (6) with the uniformly distributed stochastic reference

$$E(n_t|x_t) = 0 \tag{16}$$

Eq. (16) shows that the conditional mean of quantization noise is independent of the input x_t and that it equals zero. This is a very important result because generally without the stochastic reference, the quantization noise is dependent on the input signal x(t). F. Castanie[29] verified that the uniformly distributed stochastic reference signal bounded by $\pm\delta$ and satisfying the condition of (14) gives the minimum variance of quantization noise. In general, the benefits of the stochastic computing come from the simplicity of the computing elements.[10,15,19,20] For example, a multiplication needs only one AND gate.

Here we have shown that stochastic computing is a special case of System 1.

[2] Digital Correlator

For a long time, the correlation function estimation method using the polarity coincidence is well known.[3] [31]

One can calculate the correlation function of x(t) from that of y(t) in the system of Fig. 3. This system was used because of the calculation simplicity in y(t) as stated in Sec. I.

The mathematical analysis for the correlation

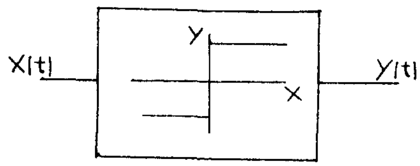


Fig. 3. One-bit quantizer.

function estimation by a polarity method using stochastic reference signals was done by H. Berndt[7] and others[8,29] but by different ways.

Consider the Eq. (11) of System 1,

$$E(Y_tY_s|x_t,x_s)= x_tx_s + E(d_s)x_t + E(d_t)x_s + E(d_td_s) \quad (17)$$

Eq. (17) was derived under the condition of LS(0, 0),

$$\Gamma d\left(\frac{\pi k}{\delta}, \frac{\pi l}{\delta}\right) = 0 \text{ for } k,l \neq 0 \quad (18)$$

If $E(d_s) = E(d_td_s) = 0$ with the condition (18), we obtain from Eq. (17)

$$E(Y_tY_s|x_t,x_s) = x_tx_s \quad (19)$$

Eq. (19) means that the correlation function can be derived from the output of quantizer with a stochastic reference and that the output correlation is the input correlation itself. If one use one bit quantizer with a stochastic reference, it becomes the digital correlator of H. Berndt[7]. It is seen that the condition $\Gamma d_1\left(\frac{\pi k}{\delta}\right) = \Gamma d_2\left(\frac{\pi k}{\delta}\right) = 0$ for $k,l \neq 0$ of Ke-Yeng Chang et al[8]'s modified digital correlator is a special case of Eq. (18).

The merit of digital correlator is in the simplicity of circuits. One can say that the digital correlator is a special case of stochastic computing.

[3] Stochastic references in speech and image coding

The first use of a stochastic reference signal in image processing was in the L.G Roberts Paper[1] because of its smoothing effect with respect to the quantization noise. After him, the stochastic reference signals were also used in low-bit rate PCM systems.[27-28]

The essential benefits of using stochastic references come from Eqs. (5)′ and (6)′ of System 2, that is,

$$E(y_t|x_t) = x_t \quad (20)$$

$$E(n_t|x_t) = 0 \quad (21)$$

$$E(n_t^2|x_t) = \frac{\delta^2}{3} \quad (22)$$

under the condition

$$\Gamma d\left(\frac{\pi k}{\delta}\right) = 0 \text{ for } k \neq 0 \quad (23)$$

Eq. (20) means that the conditional expectation of the output of System 2 is equal to the input and Eqs. (21), (22) mean that the expectation of quantization noise is equal to zero and that the quantization noise power is equal to $\frac{\delta^2}{3}$ and independent of the input signal. The independence of quantization noise to the input signal is very important for the quality of the image or the speech waveform for low-bit rate PCM.[1,22,27,28,30] Eq. (11)′ of the System 2 shows that under the condition,

$$\Gamma d\left(\frac{\pi k}{\delta}, \frac{\pi l}{\delta}\right) = 0 \text{ for } k,l \neq 0 \quad (24)$$

the correlation function of the input signal is preserved at the output of System 2, that is

$$E(y_ty_s|x_t,x_s) = x_tx_s \quad (25)$$

Eq. (25) shows that one can compute the correlation function at the output of System 2. But the simplicity of circuits does not come from System 2 like System 1, that is, one cannot use the benefits of one bit quantizer. As another application of previous results, S.S. An[30] has used Eq. (7)′ of System 2 to compute the optimal second order characteristic of the stochastic reference which gives the minimum noise after the low-pass filter attached to the output of System 2 under the condition that the input signal is first order Markov and stationary process.

IV. DISCUSSION

At Sec III, we have shown how the formulae are applied in many domains. But we have not discussed the similarity between the two systems. First we remark the similarity of the characteristic functions of the two systems. This signifies the behaviours of the two systems are very similar. We can see this fact from the conditions, Eq. (14), Eq. (18) of System 1 and Eq. (23), Eq. (24) of System 2. That is, as far as we are concerned with the first and second order moment, the conditions to give the invariance of the input signal at the output of each system are equal. (Here 'invariance' means that the input characteristic is preserved at the output.)

From Eqs. (6) and (6)$'$ of Table 1, the quantization noise power of System 1 is always larger than that of System 2. In the case of the uniformly distributed dither over the quantization step, one can easily verify that the quantization noise power ratio of the two systems is equal to 2.

Consider again Eq. (25). Eq. (25) shows that the correlation of the input is preserved at the output. This is a very important point in image processing or speech processing because it means that the spectrum of the input is preserved at the output in the respect of probability.

As far as concerned with higher order moments, one must use Eqs. (3), (3)$'$, (4), (4)$'$, (9), (9)$'$ and (10), (10)$'$. (For example, in the case of the estimation of an arbitrary function).

One can see that the m^{th} order moment of quantization noise is independent of the input signal with the condition of $d(t) \epsilon LS(m)$ for System 1 from Eq. (4). But for System 2, any order moment of quantization noise is independent of the input signal with $d(t) \epsilon LS(0)$ from Eq. (4)$'$. That is, for System 1, the condition of the independence of quantization noise upon the input is changing with higher order moments.

But for System 2, it is constant, that is, $d(t) \epsilon LS(0)$ with any order moments. For the second order properties one can derive the same conclusion from Eqs. (10) and (10)$'$. As far as concerned with higher order moments, this is the main difference between the two systems.

In the case of without the conditions of $d(t) \epsilon LS(m)$ and $d(t) \epsilon LS(m,n)$, one must directly attack Eqs. (1), (1)$'$, (2) (2)$'$, (7), (7)$'$, (8), (8)$'$. In this case it is very difficult to manipulate these equations[30].

The simplest stochastic reference satisfying the condition of LS(m) is the m+1 times convolution of a uniformly distributed one. In this case, the characteristic function of this is

$$\Gamma d(\omega) = (\mathrm{sinc}\, 2\delta f)^{m+1} \quad (26)$$

where $\omega = 2\pi f$.

One can easily verify that Eq. (26) satisfied the condition of LS(m). A little more general one is produced from Eq. (26), that is

$$\Gamma d(\omega) = (\mathrm{sinc}\, 2\delta f)^{m+1}\, G(\omega) \quad (27)$$

where $G(\omega)$ is a characteristic fcn. of an arbitrary probability desntiy and $\omega = 2\pi f$.

But it is very difficult to discuss the general class of probability density functions satisfying the condition of LS(m) as well as LS(m,n).

V. Conclusion

Two quantization systems with a stochastic reference have been analyzed. Main formulae that characterize the output signal and the quantization noise were derived.

From these formulae it was shown that the classical stochastic computing is a special case of one of the two quantization systems with a stochastic reference, that is, the system using one bit quantizer with a stochastic reference. The correlation function estimation by a polarity method using a stochastic reference signal is also shown to be a special case of the same system.

It was also shown that one can compute the correlation function at the output of the quantizer system, which was considered by L.G. Roberts.

The conditions to give the invariance of the first and second moment of the input signal at the output of each system were given and they are shown to be same for the two systems considered.

Appendix

Here we prove only Eqs. (1), (2), (3), (4), (5), (6) of System 1. The proofs of the rest formulae are very long but the method of proof is very similar.[30]

(a) Proof of (1) and (2)

At System 1, $Y_t = x_t + n_t$, where n_t is quantization noise. Therefore,

$$P_r(Y=2k\delta|x) = P_r((2k-1)\delta - x < d < (2k+1)\delta - x)$$

$$= P_r(n-\delta < d < n+\delta)$$

where we omitted the subscribts and $n = Y\text{-}x$.

The conditional characteristic function of Y_t is

$$\bar{\Gamma}_Y(\omega|x) \triangleq E[e^{-j\omega Y}|x]$$

$$\triangleq \int_{-\infty}^{\infty} e^{-j\omega Y} P_Y(Y|x)\, dY$$

$$= \sum_k \int_{-\infty}^{\infty} e^{-j\omega Y} \int_{n-\delta}^{n+\delta} P_d(m)dm\, \delta(x+n-2k\delta)dn$$

$$= \sum_k e^{-j\omega x} \int_{-\infty}^{\infty} \int_{-\delta}^{\delta} e^{-j\omega n} P_d(m+n)dm\, \delta(x+n-2k\delta)dn$$

$$= \sum_k e^{-j\omega x} \int_{-\delta}^{\delta} \int_{-\infty}^{\infty} e^{-j\omega n} P_d(m+n)\, \delta(x+n-2k\delta)dndm$$

$$= \sum_k e^{-j2k\delta\omega} \int_{-\delta}^{\delta} P_d(2k\delta - x + m)dm$$

$$= \sum_k e^{-j2k\delta\omega} \int_{-\infty}^{\infty} P_d(2k\delta - x+m) \underset{-\delta<m<\delta}{I}\, dm$$

where $\underset{-\delta<m<\delta}{I} = \begin{cases} 1 & \text{for } -\delta<m<\delta \\ 0 & \text{otherwise} \end{cases}$ and " $\triangleq$ " means a definition and $j=\sqrt{-1}$

Here applying the Parseval theorem, we obtain

$$\bar{\Gamma}_Y(\omega|x) = \sum_k e^{-j2k\delta\omega} \int_{-\infty}^{\infty} e^{j\omega' x - j\omega' 2k\delta}$$

$$\Gamma_d(\omega')2\delta \,\text{sinc}\, 2\delta f' df'$$

$$= \int_{-\infty}^{\infty} e^{j\omega' x} (\sum_k e^{-j2k\delta(\omega+\omega')}) \Gamma_d(\omega')2\delta\,\text{sinc}\, 2\delta f' df'$$

$$= \int_{-\infty}^{\infty} e^{j\omega' x} (\sum_k \delta(f+f'-\frac{k}{2\delta}))\, \Gamma_d(\omega')\text{sinc} 2\delta f' df'$$

$$= \sum_k e^{-jx(\omega - \frac{\pi k}{\delta})}\, \Gamma_d(-\omega + \frac{\pi k}{\delta})\, \text{sinc} 2\delta (f - \frac{k}{2\delta}) \qquad \text{(A1)}$$

where $\omega \triangleq 2\pi f$.

Eq. (A1) is Eq.(1). To obtain Eq. (5), we use

$$\bar{\Gamma}_n(n|x) \triangleq E(e^{-j\omega n}|x)$$

$$= E(e^{-j\omega Y + j\omega x}|x)$$

$$= e^{j\omega x}\, \bar{\Gamma}_Y(\omega|x) \qquad \text{(A2)}$$

Using Eq. (A1) and Eq. (A2), we derive Eq. (2).

(b) Proof of (3), (4), (5), (6)

$$(-j)^m E(Y^m|x)$$

$$= \frac{\partial}{\partial\omega}\, \bar{\Gamma}_Y(\omega|x)\Big|_{\omega=0}$$

$$= \sum_k \sum_{i=0}^{m} {}_mC_i \left(\frac{\sin(\delta\omega - \pi k)}{\delta\omega - \pi k}\right)^{(i)}\Big|_{\omega=0}$$

$$\left\{ \sum_{q=0}^{m-i} {}_{m-i}C_q\, \Gamma_d^{(q)}(\tfrac{\pi k}{\delta})(-jx)^{m-i-q} e^{j\frac{\pi k}{\delta}x} \right\} \qquad \text{(A3)}$$

Suppose that

$$\Gamma_d^{(q)}(\tfrac{\pi k}{\delta}) = 0 \text{ for } q = 0, 1, \cdots, m \text{ and } k \neq 0 \qquad \text{(A4)}$$

$$\left(\frac{\sin\delta\omega}{\delta\omega}\right)^{(i)}\Big|_{\omega=0} = \begin{cases} \dfrac{\delta^i (j)^i}{i+1} & \text{for } i: \text{even} \\ 0 & \text{for } i: \text{odd} \end{cases} \qquad \text{(A5)}$$

where $j=\sqrt{-1}$.

$$\Gamma_d^{(q)}(0) = (-j)^q E(d^q) \qquad \text{(A6)}$$

with Eq. (A3), Eq. (A4), Eq. (A5) and Eq(A6),

$$E(Y^m|x) = \sum_{i=0}^{m} \sum_{q=0}^{m-i} {}_mC_i\, {}_{m-i}C_q \frac{\delta^i}{i+1} E(d^q) x^{m-i-q}$$

$$= \sum_{i=0}^{m} \sum_{q=0}^{m-i} \frac{m!}{i!q!(m-i-q)!} \frac{\delta^i}{i+1} E(d^q)\, x^{m-i-q} \qquad i: \text{even}$$

With the condition (A4), this is Eq. (3). From this, one can derive Eq. (5). Eq. (4) can be easily derived by the same method from Eq. (2). Eq. (6) is a special case of Eq. (4).

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