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〈Original〉

Stress Distribution in an Infinite Plate Containing an Elliptical Crack (Part II)

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楕圓形 크랙을 포함하는 無限平板의 應力解析(第二報)

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抄 錄

이 論文에서는 第一報에서 유도한 제일종 Fredholm 積分方程式에 대하여 近似解를 解析的인 方法에 의하여 求하였으며 이 근사해를 사용하여 應力擴大係數(S.I.F.)와 크랙 에너지를 算出하였다. 또한 이 연구에서는 크랙境界 近處에서의 二次元應力値가 크랙끝에서 크랙에 수직한 平面內에서 定義된 座標 s와 θ의 값으로 表示될 수 있음을 보였다.

Nomenclature

P : Uniform load over the crack surface

$\beta'^2 = \dfrac{\lambda'+\mu'}{\mu'}$

λ', μ' : Lame's constants

ν' : Poisson's ratio

a : Length of semi-major axis of the ellipse

b : Length of semi-minor axis of the ellipse

$2h$: Thickness of the plate

1. Introduction

In this paper we continue the work of Part I which concerns the analysis of the stress distribution in an infinite medium of the finite thickness containing an elliptical crack, something that apparently has now been done before.

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Related work on the present investigation is the surface crack problem which was first considered by Irwin[2] for an infinite medium, and which, in recent years, has received certain amount of attention when the medium is of finite thickness[9,10,11]. Numerical methods, such as boundary integral method, the finite element method, and so forth, are employed to solve such problems. Notably, Smith and Sorensen[10] solved, in numerical way, the problem of a semi-elliptical crack contained in the surface of an infinite plate to which the present method is well applicable.

In Part I we have shown that the present problem is equivalent to the solution of a Fredholm integral equation of the first kind. In Part II, we consider the solution of this integral equation, i.e., Eqn. (2.23) of Part I and other quantities of physical interest. Approximate solution to the integral equation is obtained analytically and using this solution, trelevant quantities are obtained.

The primary goal of this paper is to compute the stress intensity factor at the crack border. The k_I-factor for the infinite medium derived by Irwin[2] using the solution by Green and Sneddon[1] is given by

$$k_1=\frac{P}{E(e)}\left(\frac{b}{a}\right)^{\frac{1}{2}}(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{1}{4}},$$
$$a>b \qquad (1.1)$$

where $E(e)$ is the complete elliptic integral of he second kind with argument

$$e=\left[1-(b/a)^2\right]^{\frac{1}{2}}$$

It is shown that the stress intensity factor for the present problem reduces to Eqn. (1.1) as the thickness approaches to infinity.

2. Solution of Integral Equation and Quantities of Physical Interest.

A good starting point would be the following integral equation which was derived in Part I.

$$\int_{\Omega}A(u,v)\left[\frac{1}{r^3}-\int_0^{\infty}\zeta^2M(\zeta h)J_0(\zeta r)d\zeta\right]dudv$$
$$=-\frac{P}{2(\beta'^2-1)}, \qquad (2.1)$$

where Ω is the region governed by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1.$$
$$r=[(x-u)^2+(y-v)^2]^{1/2},$$
$$M(u)=\frac{-2u(u+1)+e^{-2u}-1}{2u+\sin 2u},$$

and J_0 is the Bessel function in the usual notation.

By letting $\zeta h=\eta$, we may write the second term in the bracket on the left of Eqn. (2.1) as

$$h^{-3}\int_0^{\infty}\eta^2M(\eta)J_0\left(\frac{\eta}{h}r\right)d\eta.$$

By expanding the Bessel function in a series expansion, we find that the above expression is equal to

$$\frac{I_1}{h^3}-\frac{r^2}{4h^5}I^2+\frac{r^4}{64h^7}I_3+O(h^{-9}), \qquad (2.2)$$

where

$$I_n=\int_0^{\infty}\eta^{2n}M(\eta)d\eta. \qquad (2.3)$$

The integral appearing in Eqn. (2.3) has been tabulated by Ling[5].

Substituting (2.2) into Eqn. (2.1), we see that $A(u,v)$ must be of the form

$$A(u,v)=A_0(u,v)+h^{-3}A_1(u,v)$$
$$+h^{-5}A_2(u,v)+0(h^{-6}), \qquad (2.4)$$

where

$$\int_{\Omega}\frac{A_0(u,v)}{r^3}dudv=-\frac{P}{2(\beta'^2-1)}, \qquad (2.5)$$

$$\int_{\Omega}\frac{A_1(u,v)}{r^3}dudv=I_1\int_{\Omega}A_0(u,v)dudv, \quad (2.6)$$

$$\int_{\Omega}\frac{A_2(u,v)}{r^3}dudv=-\frac{I_2}{4}\int_{\Omega}A_0(u,v)r^2\,dudv.$$
$$(2.7)$$

The integrals on the left of the above equations are to be considered as

$$\int_{\Omega}\frac{(\cdot\cdot)}{r^3}dudv=\nabla_1{}^2\int_{\Omega}\frac{(\cdot\cdot)}{r}dudv,$$

where

$$\nabla_1{}^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

The solutions to Eqns. (2.5), and (2.6) are, from section 4 of Part I easily found to be

$$A_0(x,y)=\frac{bP}{4\pi E(e)(\beta'^2-1)}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}},$$
$$(x,y)\in\Omega \qquad (2.8)$$

$$A_1(x,y)=-\frac{I_1ab_2}{3E(e)}A_0(x,y),$$
$$(x,y)\in\Omega \qquad (2.9)$$

Eqn. (2.8) represents the solution for an infinite medium containing an elliptical crack subjected to uniform normal pressure.

The solution to Eqn. (2.7) can also be obtained by straightforward calculations; see, for example, Lur'e[6]. $A_2(x,y)$ is found to be

$$A_2(x,y)=A_0(x,y)(\alpha x^2+\beta y^2+\gamma),$$
$$(x,y)\in\Omega \qquad (2.10)$$

where α, β and γ are constants involving complete elliptic integrals of the first and second kind, and the detailed expressions are listed in Appendix I. We can obtain to any higher order of h we desire by simply carrying out more expansions.

Hence the displacements and stresses at any point of the elastic plate under the consideration can be obtained from Eqns. (2.6), (2.7), and (2.8) of Part I. Of immediate interest is the normal stress component σ_z outside the ellipse on the plane $z=0$. It is now more pertinent to examine the normal stress σ_z near the border of the elliptical crack. As the distance normal to the crack border tends to zero, other terms in the expression of normal stress σ_z remain finite except the integral

$$\int_\Omega \frac{A(u,v)}{r^3} dudv$$

To obtain the stress intensity factor, we first evaluate the integral

$$\int_\Omega \frac{\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{1/2}}{r^3} dudv \tag{2.11}$$

subjected to the condition that the point(x,y) is located outside of Ω, in the vicinity of crack edge. We now introduce polar coordinates ρ, λ with(x,y) as origin:

$$u-x=\rho\cos\lambda,\quad v-y=\rho\sin\lambda \tag{2.12}$$

By the transformation formulae(2.12)

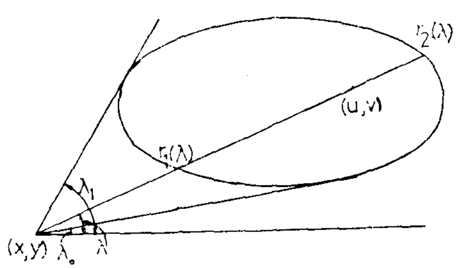

Fig. 1

$$1-\frac{u^2}{a^2}-\frac{v^2}{b^2}=L-2\rho\Phi_1-\rho^2\Phi_2 \tag{2.13}$$

where

$$L=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$\Phi_1=\frac{x\cos\lambda}{a^2}+\frac{y\sin\lambda}{b^2},$$

$$\Phi_2=\frac{\cos^2\lambda}{a^2}+\frac{\sin^2\lambda}{b^2}$$

With these transformations, the expression (2.11) assumes the form

$$\int_{\lambda_0}^{\lambda_1}\int_{r_1(\lambda)}^{r_2(\lambda)} \frac{(L-2\rho\Phi_1-\rho^2\Phi_2)^{1/2}}{\rho^2} d\rho d\lambda \tag{2.14}$$

where $r_1(\lambda)$ and $r_2(\lambda)$ as well as the limits of integration with respect to λ are shown in Fig. 1.

If we integrate the inner integral of expression (2.14) by parts, noticing that the integrand vanishes at both limits, the inner integral becomes

$$\int_{r_1(\lambda)}^{r_2(\lambda)} \frac{(\Phi_1+\Phi_2\rho)d\rho}{\rho[L-2\rho\Phi_1-\rho^2\Phi_2]^{1/2}} \tag{2.15}$$

Setting $\sqrt{\frac{\rho-r_1}{r_2-\rho}}=t$, integrals in the above are elementary, and expression(2.15) is equal to

$$-\pi\left[\frac{\Phi_1}{\sqrt{\Phi_2 r_1(\lambda)r_2(\lambda)}}+\frac{1}{\sqrt{\Phi_2}}\right]$$

Finally, since $r_1(\lambda)r_2(\lambda)=-\frac{L}{\Phi_2}$, integral (2.11) now takes the form

$$-\pi\int_{\lambda_0}^{\lambda_1}\left\{\frac{\Phi_1}{\sqrt{-L}}+\frac{1}{\sqrt{\Phi_2}}\right\}d\lambda \tag{2.16}$$

The second term in expression(2.16) remains finite as the distance normal to the crack edge approaches zero so we direct our attention to the first term. Performing the integration, and substituting the value of $\sin\lambda_0$, and so forth, from Appendix II, the first term of(2.16) yields

$$\frac{2\pi}{ab\sqrt{-L}}(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{1}{2}} +0(\sqrt{s}), \tag{2.17}$$

where, referring to Fig. 2, s is the radial distance from the crack front.

It can be easily shown that

$$-L=\frac{2s}{ab}(b\cos\phi\ \cos\omega+a\sin\phi\ \sin\omega) +0(\sqrt{s}), \tag{2.18}$$

where ω is the angle between the outward unit normal vector of the crack border and the x-axis (Fig. 3 of Appendix Ⅱ), and is related to the parametric equation of an ellipse as

$$\left.\begin{aligned} x'&=a\sin\phi=(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{1}{2}}\sin\omega \\ y'&=b\cos\phi=(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{1}{2}}\cos\omega \end{aligned}\right\} \tag{2.19}$$

From (2.17), and Eqns. (2.18) and (2.19) the integral (2.11) has the form

$$\frac{2\pi}{\sqrt{2abs}}(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{1}{4}}$$

where we have omitted the terms with no inverse square root singularity.

Altogether, it can be easily shown that the normal stress σ_z on the plane $z=0$ near the crack border is

$$\sigma_z=\frac{K_1}{\sqrt{2s}}+O(h^{-6})$$

The stress intensity factor K_1 in the above equation is equal to

$$K_1=\frac{P}{E(e)}\left(\frac{b}{a}\right)^{1/2}(a^2\sin^2\phi+b^2\cos^2\phi)^{1/4}\left[1-\frac{I_1ab^2}{3E(e)h^3}+\frac{H(\phi)}{h^5}\right], \tag{2.20}$$

where

$$H(\phi)=\alpha\, a^2\sin^2\phi+\beta\, b^2\cos^2\phi-\gamma$$

As the thickness of the plate tends to infinity Eqn. (2.20) agrees with what was obtained by Irwin(2).

Another expression of physical interest is the energy W required to open out the crack. This is given by the equation

$$W=\mu' P\int_\Omega u_z\,dudv \tag{2.21}$$

Putting the expression of u_z into above equation, we obtain

$$W=\frac{\mu'(1-\nu')2\pi a b^2 P^2}{3E(e)}\left[1-\frac{ab^2I_1}{E(e)h^3}\right]+O(h^{-5})$$

3. Stress Distribution Near Crack Border

In this section we study the stress distribution near the border of the elliptical crack to acquire better knowledge of current three dimensional fracture mechanics theory. For this, it is expedient to use the following elliptic integral as used in Seo and Mura(7)

$$\int_{\Omega_1}\frac{dudvdw}{[(x-u)^2+(y-v)^2+(z-w)^2]^{\frac{1}{2}}}=\pi abc\int_\lambda^\infty\frac{U}{\Delta}ds, \tag{3.1}$$

where

$$\Omega_1:\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1,$$

$$U=1-\left(\frac{y^2}{a^2+s}+\frac{y^2}{b^2+s}+\frac{z^2}{c^2+s}\right),$$

$$\Delta=\sqrt{(a^2+s)(b^2+s)(c^2+s)}$$

From Eqn. (3.1), we obtain

$$\Psi=\int_\Omega\frac{\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{\frac{1}{2}}dudv}{[(x-u)^2+(y-v)^2+z^2]^{\frac{1}{2}}}=\frac{\pi ab}{2}\int_\lambda^\infty\left(1-\frac{x^2}{a^2+s}-\frac{y^2}{b^2+s}-\frac{z^2}{s}\right)\frac{ds}{\sqrt{Q(s)}}, \tag{3.2}$$

where $Q(s)=(a^2+s)(b^2+s)s$.

The analysis in the following would be more convenient by application of ellipsoidal coordinates (λ,μ,ν) in terms of which the coordinates (x,y,z) at any point can be expressed as(8)

$$\left.\begin{aligned} a^2(a^2-b^2)x^2&=(a^2+\lambda)(a^2+\mu)(a^2+\nu) \\ b^2(b^2-a^2)y^2&=(b^2+\lambda)(b^2+\mu)(b^2+\nu) \\ a^2b^2z^2&=\lambda\mu\nu \end{aligned}\right\} \tag{3.3}$$

where

$$-a^2\leqslant\nu\leqslant-b^2\leqslant\mu\leqslant 0\leqslant\lambda<\infty,$$

when $z=0$, $\lambda=0$ represents the point (x,y) inside the ellipse and $\mu=0$ outside the ellipse.

By writing λ in terms of Jacobian elliptic functions

$$\lambda=a^2\,\mathrm{cn}^2 u/\mathrm{sn}^2 u=a^2(\mathrm{sn}^{-2}u-1),$$

the second derivative of Ψ is given as[1]

$$\frac{\partial^2\Psi}{\partial z^2}$$

$$=\frac{2\lambda^{\frac{1}{2}}[\lambda(a^2b^2-\mu\nu)-a^2b^2(\mu+\nu)-(a^2+b^2)\mu\nu]}{a^2b^2(\lambda-\mu)(\lambda-\nu)(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}}$$

$$-\frac{2}{ab^2}\left[E(u)-\frac{snu\ cnu}{dnu}\right] \tag{3.4}$$

where $E(u)=\int_0^u dn^2t\ dt.$

Kassir and Sih[3] have shown that, in the vicinity of crack, λ, μ, and ν can be expressed as

$$\left.\begin{aligned}\lambda&=\frac{2\,abs}{(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{1}{2}}}\left(\cos\frac{\theta}{2}\right)^2\\ \mu&=-\frac{2\,abs}{(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{1}{2}}}\left(\sin\frac{\theta}{2}\right)^2\\ \nu&=-(a^2\sin^2\phi+b^2\cos^2\phi)\end{aligned}\right\} \tag{3.5}$$

s and θ in the above equations are, as shown in Fig.2, radial distance from the crack front, and angle between the plane of the crack and s, respectively. Both s and θ lie in a plane normal to the border.

Thus if we use Eqns. (3.4), (3.5) and the third of Eqn. (2.7) of Part I, we obtain the stress intensity factor K_1

$$K_1=\frac{P}{E(e)}\left(\frac{b}{a}\right)^{\frac{1}{2}}(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{1}{4}}$$

$$\left\{1-\frac{I_1ab^2}{3E(e)h^3}\right\} \tag{3.6}$$

which is in agreement with Eqn. (2.21) up to the second term.

The derivatives of Eqn. (3.2) are listed in[3]. If we make use of these expressions and substitute Eqn. (3.5) into Eqns. (2.7) and (2.8)

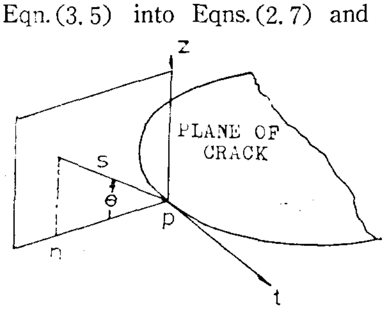

Fig. 2 Coordinates s and θ at the edge of the crack.

of Part I, the stress distribution in the immediate vicinity of the crack border is found. Thus, for instance, σ_z is as follows.

$$\sigma_z=\frac{K_1}{\sqrt{2s}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)+0(h^{-5})$$

where K_1 is given by Eqn. (3.6).

4. Solution for the Penny-shaped Crack

In this section we consider the iterative solution of the integral equation (3.7) of Part I for the penny-shaped crack. It will as well serve a good comparison to the solution for the elliptical crack obtained in section 2 when the ellipse reduces to a circle. Before doing this, we note some errors in section 3 of Part I. σ is to be multiplied to $A(\sigma)$ throughout though the analysis is correct, and the free term is incorrect. The correct form of the integral equation is as follows.

$$g(t)+\int_0^a g(\rho)K(\rho,t)d\rho$$

$$=-\frac{Pt^2}{8\pi(\beta'^2-1)}+C \tag{4.1}$$

where $K(\rho,t)=\int_0^\infty M(\zeta h)\cos t\zeta\ \cos\rho\zeta d\zeta$

In Eqn. (4.1), C is a constant of integration which can be determined from the condition

$$g(t)=0 \text{ when } t=a \tag{4.2}$$

The displacement over the crack surface is given by the equation

$$A(\sigma)=-\frac{2}{\pi}\ \frac{d}{\sigma d\sigma}\int_\sigma^a\frac{tg(t)dt}{\sqrt{t^2-\sigma^2}}$$

$$=\frac{2}{\pi}\ \frac{d}{\sigma d\sigma}\int_\sigma^a g(t)\sqrt{t^2-\sigma^2}\ dt. \tag{4.3}$$

Instead of determining the constant C in Eqn. (4.1) and solving the integral equation, it is far better to eliminate it by simply differentiating the equation with respect to t and then integrate by part in the second of the left-hand side using the condition (4.2) to obtain the following alternative form, and use the last equality of Eqn. (4.3) to evaluate the displacement over the crack.

$$g'(t)-\frac{2}{\pi}\int_0^a g'(\rho)K'(\rho,t)d\rho$$

$$=-\frac{Pt}{4\pi(\beta'^2-1)}, \qquad (4.4)$$

where

$$K'(\rho,t)=\int_0^\infty M(\zeta h)\ \sin t\zeta \sin \rho\zeta d\zeta$$

If we let $\eta=\zeta h$ and expand the sine function in the power series, we obtain

$$K'(\rho,t)=t\Big[\frac{I_1\rho}{h^3}-\frac{I_2(\rho^3+\rho t^2)}{h^5 3!}\Big]+0(h^{-6})$$

By standard iteration procedure[12] the solution to (4.4) is given by

$$g'(t)=-\frac{P}{4\pi(\beta'^2-1)}\Big[t-\frac{a^3 2I_1}{3\pi h^3}t+\frac{2}{\pi}$$

$$\frac{I_2}{h^5 3!}\Big(\frac{a^5}{5}t+\frac{a^3}{3}t^3\Big)\Big]+0(h^{-6}) \qquad (4.5)$$

Substituting from Eqn. (4.5) into Eqn. (4.3), we obtain the displacement over the penny-shaped crack

$$A(\sigma)=\frac{P}{2\pi(\beta'^2-1)}(a^2-\sigma^2)^{\frac{1}{2}}\Big[1-\frac{a2I_1}{3\pi h^3}$$

$$+\frac{I_2}{\pi h^5}\Big(\frac{14a^5}{135}+\frac{2}{27}a^3\sigma^2\Big)\Big]+0(h^{-6}).$$

If we compare the above equation with that given by Eqn. (A-7) of Appendix I we find that the agreement between two solutions is complete as the ellipse reduces to a circle.

5. Conclusion

In this paper, we have, theoretically, derived the approximate values of the stress intensity factor and the normal stress σ_z near the border of an elliptical crack in an isotropic plate. Since the values of I_1 and I_2 are -4.230 and -29.817, respectively, we see that, when the ellipse reduces to a circle, the percentage increase in the stress intensity factor from that for the infinite medium is about 10% with the accuracy of order of h^{-5} when $h/a=2$.

Appendix I

To find α, β, and γ, we need the values of following two integrals which were evaluated by transforming the coordinates as in section 2 and by the similar analysis

$$\int_\Omega \frac{u\sqrt[3]{L}}{r^3}dudv=\pi\int_0^{\frac{\pi}{2}}\frac{\cos^2\lambda d\lambda}{\sqrt{\Phi_2}}+\frac{\pi x^2}{a^2}$$

$$\Big[10\int_0^{\frac{\pi}{2}}\frac{d\lambda}{\sqrt{\Phi_2}}+\frac{1}{b^2}\int_0^{\frac{\pi}{2}}\frac{\cos^2\lambda\sin^2\lambda d\lambda}{\Phi_2\sqrt{\Phi_2}}$$

$$+\Big(\frac{10}{b^2}-\frac{22}{a^2}\Big)\int_0^{\frac{\pi}{2}}\frac{\cos^2\lambda d\lambda}{\Phi_2\sqrt{\Phi_2}}+12\Big(\frac{1}{a^4}$$

$$-\frac{1}{a^2b^2}\Big)\int_0^{\frac{\pi}{2}}\frac{\cos^4\lambda d\lambda}{\Phi_2^2\sqrt{\Phi_2}}-\frac{\pi y^2}{a^2b^2}$$

$$\int_0^{\frac{\pi}{2}}\frac{\cos^4\lambda d\lambda}{\Phi_2\sqrt{\Phi_2}} \qquad \text{(A-1)}$$

$$\int_\Omega \frac{v^2\sqrt{L}}{r^3}dudv=\pi\int_0^{\frac{\pi}{2}}\frac{\sin^2\lambda d\lambda}{\sqrt{\Phi_2}}-\frac{\pi x^2}{a^2b^2}$$

$$\int_0^{\frac{\pi}{2}}\frac{\sin^4\lambda d\lambda}{\Phi_2\sqrt{\Phi_2}}-\frac{\pi y^2}{b^2}\Big[10\int_0^{\frac{\pi}{2}}\frac{d\lambda}{\sqrt{\Phi_2}}+\frac{1}{a^2}$$

$$\int_0^{\frac{\pi}{2}}\frac{\cos^2\lambda\sin^2\lambda d\lambda}{\Phi_2\sqrt{\Phi_2}}+\Big(\frac{10}{a^2}-\frac{22}{b^2}\Big)$$

$$\int_0^{\frac{\pi}{2}}\frac{\sin^2\lambda d\lambda}{\Phi_2\sqrt{\Phi_2}}+12\Big(\frac{1}{b^4}-\frac{1}{a^2b^2}\Big)$$

$$\int_0^{\frac{\pi}{2}}\frac{\sin^4\lambda d\lambda}{\Phi_2^2\sqrt{\Phi_2}}\Big] \qquad \text{(A-2)}$$

where $L=1-\frac{u^2}{a^2}-\frac{v^2}{b^2}$, and Φ_2 is as in Eqn. (2.13). In the evaluation of the above integrals $\int_\Omega \frac{(\cdot\cdot)}{r^3}dudv$ is considered as in Eqns. (2.4)–(2.6) whenever necessary. After reduction of the above integrals in the standard Legendre form, one finds that the right-hand side of Eqn. (A-1) is equal to

$$\frac{\pi b}{e^2}\Big[K(e)-E(e)-\frac{x^2}{a^2}\Big\{2\frac{E(e)}{1-e^2}$$

$$-2\frac{E(e)-K(e)}{e^2}+3K(e)-6E(e)\Big\}$$

$$-\frac{y^2}{a^2}\Big\{\frac{E(e)-K(e)}{e^2}+\frac{E(e)}{1-e^2}\Big\}\Big]$$

and one obtains

$$\frac{\pi b}{e^2}\Big[E(e)-(1-e^2)K(e)-\frac{x^2}{a^2}$$

$$\left\{2\frac{E(e)-K(e)}{e^2}-E(e)+2K(e)\right\}$$
$$-\frac{y^2}{a^2}\left\{\frac{6E(e)}{1-e^2}-2\frac{E(e)-K(e)}{e^2}\right.$$
$$\left.-5K(e)-2E(e)\right\}\Big]$$

for the right-hand side of Eqn. (A-2). In the above, $E(e)$, $K(e)$ are the complete elliptic integrals of the first and second kind, respectively.

Another equation needed is the following which can be obtained from the integral used in section 4 of Part I

$$\int_\Omega \frac{\sqrt{L}}{r^3}dudv=-\frac{2\pi E(e)}{b}$$

Thus we have three equations to determine three unknowns.

Now, from what precedes, the evaluation of the values of α,β, and γ is immediate. They are found to be

$$\alpha=\frac{a^3}{e^2}\left[4\frac{E(e)-K(e)}{e^2}-\frac{6E(e)}{1-e^2}\right.$$
$$\left.-E(e)-7K(e)\right]I_2/6D, \tag{A-3}$$

$$\beta=\frac{a^3}{e^2}\left[4\frac{E(e)-K(e)}{e^2}-\frac{E(e)}{1-e^2}\right.$$
$$\left.+6E(e)-3K(e)\right]I_2/6D, \tag{A-4}$$

$$\gamma=\frac{a^5(1-e^2)I_2}{12E(e)}\left[\frac{2-e^2}{5}-\frac{N}{D}\right], \tag{A-5}$$

where

$$D=\frac{E(24K-57E)}{e^4(1-e^2)}+\frac{12E^2}{e^4(1-e^2)}$$
$$+\frac{1}{e^4}(12E^2+24EK-15K^2),$$

and

$$N=\frac{K-E}{e^2}\left(\frac{-5E}{e^2(1-e^2)}+\frac{10K}{e^2}+\frac{7E}{e^2}\right)$$
$$-K\left(\frac{E}{e^2(1-e^2)}+\frac{3K}{e^2}-\frac{6E}{e^2}\right),$$

and K, E are shorthand notation for $K(e)$ and $E(e)$, respectively.

Of interest is the case when the ellipse reduces to a circle. To calculate the limiting values of α, β, and γ as $e\to 0$, care has to be taken. For instance, to evaluate the limit of D, we first rearrange it to obtain

$$D=\left(\frac{E}{e^2(1-e^2)}-\frac{K}{e^2}\right)\left\{\frac{30K}{e^2}-\frac{39E}{e^2}\right.$$
$$\left.+12\left(\frac{E}{e^2(1-e^2)}-\frac{K}{e^2}\right)\right\} \tag{A-6}$$
$$+\frac{3K^2}{e^4}+\frac{12E^2}{e^4}-\frac{15EK}{e^4}$$
$$+\frac{18(E-K)}{e^4}\left\{\frac{E-K}{e^2}\right.$$
$$\left.-\left(\frac{E}{e^2(1-e^2)}-\frac{K}{e^2}\right)\right\}$$

The limit of each term grouped by the parenthesis as $e\to 0$ is now easily obtained using the definitions of the complete elliptic integrals and the derivatives of them along with the L'hospital's rule. The limit of the term in the first parenthesis is $\frac{\pi}{4}$, $K/e^2\to -E/e^2\to\pi/8$, $\frac{E-K}{e^4}\to\frac{3\pi}{32}$ as $e\to 0$, thus, finally, $D\to\left(\frac{\pi}{8}\right)^2 270$. In a similar manner, we can show that $N\to\left(\frac{\pi}{8}\right)^2 60$. The numerators of Eqn. (A-3) and (A-4) approach to $-30\pi/16$. Thus, altogether, we have

$$\alpha=-\frac{2a^3I_2}{27},\quad \beta=-\frac{2a^3I_2}{27},$$
$$\gamma=-\frac{14a^5I_2}{135} \tag{A-7}$$

In actual numerical computations when e is extremely small, Eqn. (A-3)–(A-5) may not give correct values. In such case it is better to use Eqn. (A-6) and, so forth, and expand in terms of e in Taylor series and then truncate appropriately.

Appendix II

For the analysis of stresses on the plane $z=0$ near the crack border, the values of $\sin\lambda_0$, etc., are to be expressed in terms of the radial distance to the crack border. In Fig.3, the length PP' represents the projection of radial distance s in the binormal. It is observed from Fig.3 that the coordinate of the point $P'(x,y)$ can be expressed by

$$x=a\cos\phi+s\cos\theta\cos\omega$$
$$y=b\sin\phi+s\cos\theta\sin\omega$$

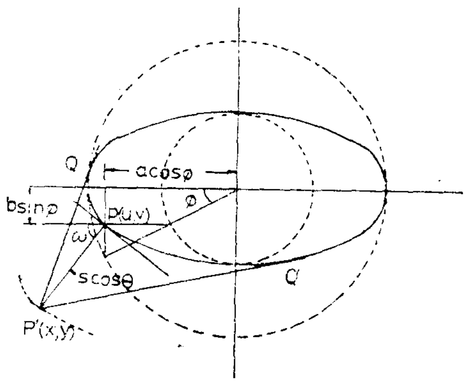

Fig. 3 Polar coordinates near crack edge.

where θ and s are shown in Fig. 2. The coordinates of points of tangency of the tangential lines from the point P to the ellipse are obtained from the formulae

$$u=\frac{x\pm\frac{y}{2b}\sqrt{-L}}{1-L},\quad v=b\sqrt{1-\frac{u^2}{a^2}}$$

where L is given in Eqn.(2.13). Expanding the quantities under the square root by using binomial expansion, we find after some manipulation, for $\theta=0$

$$\sin\lambda_0=\frac{b\cos\phi}{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}}+0(\sqrt{s})$$
$$\sin\lambda_1=-\sin\lambda_0+0(\sqrt{s})$$
$$\cos\lambda_0=-\frac{a\sin\phi}{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}}+0(\sqrt{s})$$
$$\cos\lambda_1=-\cos\lambda_0+0(\sqrt{s})$$

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