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Finite Element Analysis of Laminar Combined Free and Forced Convection through a Duct with Peripheral Heat Flux Variation.

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壁에서의 熱플럭스가 非均一한 덕트內 層流組合對流熱傳達에 對한 有限要素法에 依한 解析

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抄 錄

任意의 幾何學的形態 斷面을 가진 垂直덕트內 層流組合對流熱傳達에서 壁溫一定인 境界條件에 對하여는 Nayak[12], Giuidice[13]의 有限要素法解가 있다.

本 硏究는 境界條件으로 壁에서의 熱플럭스가 非均一(均一한 경우 包含)한 경우를 對象으로 有限要素法에 依한 解析을 試圖한 것으로 例示에서 보는 바와 같이 數値解는 嚴解密와 좋은 一致를 보이고 있다.

Nomenclature

A : Cross-sectional area of the duct

$\{B^e\}$: Element column matrix defined by equation (19. a)

c : Axial temperature gradient

C_1 : Quantity defined by $C_1=0.25G(x,y)$

c_p : Specific heat at constant pressure

d : Equivalent hydraulic diameter

$[F^e]$: Element matrix defined by equation (19)

g : Gravitational acceleration

K_f : Thermal conductivity of the fluid

L : Pressure gradient parameter, $-d^2\times(dp^*/dz^*+\rho_w{}^*g)/\mu w_m$

N_i, N_j : Piecewise linear interpolation function

n_x, n_y : Direction cosine of the outward normal

p : pressure

P : Perimeter of the duct

Q^e : Element column matrix defined by equation (19)

$\dot{q}''$: (Peripheral surface) heat flux

$\dot{q}_0''$: Average (peripheral surface) heat flux

Ra : Rayleigh number

T^*, t : Temperature

t_m : mean temperature

$\{t^e\}$: Element column matrix defined by equation (12)

Δt_{av} : Average value of Δt

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w : Velocity in z-direction
$w_m{}^*$: Mean velocity
$\{w^e\}$: Element column matrix defined by equation (13)
x, y, z : Cartesian coordinates
β : Volume expanison ratio
μ : Viscosity
ρ : Density
$\Omega^{(e)}$: Area of the triangular element
Σ : Bonudary

Subscript

w : On the wall
c : On the boundary
m : Mean value

Superscript

$*$: Dimensional quantity
e : Quantities associated with a particular triangular element

1. Introduction

The problem of combined free and forced convection through a vertical duct, under the condition of constant heat input in axial direction, has important applications in the compact heat exchangers and the nuclear reactors. Especially, the boundary condition of peripherally varying heat flux has been experienced in many practical engineering problems.

If the heat flux in the axial direction is constant, the wall temperature varies linearly along the flow direction. Hence, for fluids with constant properties, the temperature profile does not change along the axial direction. For such problems the boundary conditions can be classified into four categories, namely, (a) uniform peripheral wall temperature, (b) uniform peripheral heat flux, (c) peripherally varying heat flux and (d) peripherally varying wall temperature.

Ostrach [1] found the exact solution to the problem of parallel plate channels and Han [2] found the exact solution for the rectangular duct, both by transforming governing eguations to 4th order partial differential equations. Han [3] obtained the solution by using Fourier series. And Tao [4] used complex function to obtain the solution for the rectangular duct. For vertical circular duct, Morton [5] employed Bessel function, but Lu [6] employed finite sine transform to find the same solution. Aggarwalar and Igbal [7] used membrane analogy to obtain the solution for the triangular duct and Igbal, Ansari and Aggarwalar [8] used point matching method which is expressed in terms of Bessel function to obtain the solution for the polygonal ducts. All of the above analyses [1]-[8] were derived for the combined free and forced convection through a vertical duct with simple geometric shapes under the boundary conditions (a) and (b).

By incorporating variational method into inhomogeneous Helmholtz formular, Agrawal [9] was able to find the exact solution for the rectangular ducts. For the triangular ducts, Igbal, Aggarwalar and Fowler [10] used variational method. For non-circular ducts Igbal, Aggarwala and Khatry [11] had examined, the effect of wall thickness and used variational method to solve the problem for the boundary condition (a) and (b) with additional condition, namely, peripheral heat conduction.

Nayak and Ping Cheng [12] applied FEM in order to obtain an approximate solution to the combined free and forced convection problem with arbitrary shaped ducts.

By transforming the temperature and the velocity in terms of complex functions, Giudice, Comini and Mikhailov [13] were able to obtain the solution by FEM using Galerkin method. The analyses done in Ref. [12] and [13] were forcused on the simple boundary condition (a).

Although Reynolds [14] and Kays [15] found the solutions to the circular duct problem with the peripheral heat flux, $\dot{q}''=\dot{q}_0''(1+\cos\theta)$, it is applicable only to the forced convection, not to the combined free and forced convection.

So far, analytical solutions have been obtained for the ducts with simple geometric shapes, and FEM solutions have been available only to the problem of boundary condition (a). Therefore it is intended in this study to find the soluion by FEM to the problem of combined free and forced convection for ducts with arbitray shapes, and boundary condition (b) or (c).

2. Governing Equations and Boundary Conditions

By using finite element method the problem of combined free and forced convection for ducts with arbitrary shape is to be studied for the fully developed laminar flow. With the constant axial heat flux and fully developed temperature profile, the peripheral wall heat flux condition is either constant or varying. It is assumed in the analysis that the flow is only in vertical direction and all the fluid properties are constant except the density which varies linearly with temperature.

Additional assumptions are: the friction heat transfer of viscous flow is negligible and the internal heat generation is none.

By resorting to Ref. (3), (10), (12) the momentum and energy equations can be established as follows.

$$\mu\left(\frac{\partial^2\omega^*}{\partial x^{*2}}+\frac{\partial^2\omega^*}{\partial y^{*2}}\right)=\frac{dp^*}{dz^*}+\rho_w^*(1-\beta t^*) \quad (1)$$

$$K_f\left(\frac{\partial^2 t^*}{\partial x^{*2}}+\frac{\partial^2 t^*}{\partial y^{*2}}\right)=\rho c_p c\omega^* \quad (2)$$

The boundary condition requires that the velocity of the fluid at the wall is zero due to fluid viscosity, and the thermal boundary condition requires that

$$K_f\frac{\partial t^*}{\partial n^*}=f^*(x^*,y^*)$$

where $f^*(x^*,y^*)$ is an arbitrary function and

$$\frac{\partial}{\partial n^*}=\frac{\partial}{\partial x^*}n_x+\frac{\partial}{\partial y^*}n_y$$

To simplify the problem the following dimensionless parameters are used:

$$x=\frac{x^*}{d},\quad y=\frac{y^*}{d},\quad w=\frac{w^*}{w_m^*}$$

$$t=t^*/\rho^* c_p c d w_m^*/K_f$$

where w_m^* is defined as

$$\omega^*=\frac{1}{A}\iint w^* dxdy$$

By substituting these dimensionless parameters into equation (1) and (2) the simpler form of governing equations are obtained.

$$\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+R_a t+L=0 \quad (3)$$

$$\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}-w=0 \quad (4)$$

Even though the dimensionless boundary condition for the fluid flow at the wall can be simply expressed as $w=0$, the dimensionless boundary condition for the heat transfer, on the other hand, requires closer examination.

Igbal, Ansari and Aggarwala [8] used $\partial t/\partial n=0.25$ for their analysis of the dimensionless boundary condition (b) · In this study, however, an attempt has been made to derive the thermal boundary condition by incorporating the law of conservation of energy.

In an arbitrary shaped duct, the uniform peripheral heat flux is

$$K_f\frac{\partial t^*}{\partial n^*}=q_0''=\text{constant} \quad (5)$$

which by employing dimensionless parameters becomes

$$\frac{\partial t}{\partial n}=\frac{\dot{q}_0''}{\rho^* c_p c w_m^* d} \quad \text{at the boundary} \quad (6)$$

By the law of conservation of energy, the

Fig. 1 [15] shows that

$$\dot{q}_0'' = \frac{A\rho^* c_p w_m^* c}{P} \tag{7}$$

Also, inserting equation (7) into equation (6)

$$\frac{\partial t}{\partial n} = \frac{A}{Pd} \quad \text{at the boundary} \tag{8}$$

Since the equivalent hydraulic diameter is $d=4A/P$, the dimensionless thermal boundary condition finally becomes

$$\frac{\partial t}{\partial n} = 0.25$$

This is the same value derived in Ref. [8]

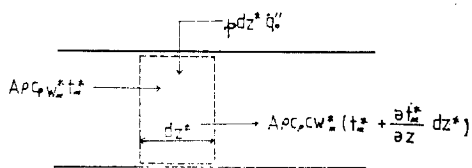

Fig. 1 Energy balance on a differential control volume

For the boundary condition of peripheral heat flux variation, the numerical value obtained above can be expressed as a constant function. For example, the average heat flux, $\dot{q}_0''$, for the ducts of peripheral heat flux can be obtained and this average can be placed as a constant for the expression that allows x and y coordinates as independent variables. Showing this mathmatically,

$$\frac{\partial t^*}{\partial n^*} = f^*(x_c^*, y_c^*) \text{ changes to}$$

$$\frac{\partial t^*}{\partial n^*} = \dot{q}_0'' f(x_c^*, y_c^*)$$

And the boundary condition (c) which is to be used in this paper for circular duct changes to

$$\frac{\partial t}{\partial n} = 0.25G(x_c, y_c)$$

3. Finite Element Formulation

The purpose of applying finite element method in this study is to obtain approximate numerical solutions to the problems of arbitrary shaped ducts and of arbitrary thermal boundary conditions. For these problems, obtaining the analytical solutions is almost an impossible task. Herbner [16] and Zienkiewicz [17] have presented a thorough study of general explanation and application of finite element method.

With the triangular elements, the unknown temperature and the velocity can be approximated in the finite element domain by the following relation;

$$t^{(e)}(x,y) = \sum_{i=1}^{3} N_i(x,y)t_i = \lceil N \rfloor \{t^e\} \tag{10}$$

$$w^{(e)}(x,y) = \sum_{i=1}^{3} N_i(x,y)w_i = \lceil N \rfloor \{\omega^e\} \tag{11}$$

where the piecewise linear linterpolation function

$$N_i = \frac{1}{2A(e)}(a_i + b_i x + c_i y) \text{ and}$$

$$a_i = x_j y_k - x_k y_j, \quad b_i = y_j - y_k, \quad c_i = x_k - x_j$$

For equations (10) and (11), using the nodal values at the vertices of the triangular element, $\{t^e\}$ and $\{w^e\}$ can be expressed in column matrices as

$$\{t^e\} = \begin{Bmatrix} t_i \\ t_j \\ t_k \end{Bmatrix} \tag{12}$$

$$\{w^e\} = \begin{Bmatrix} \omega_i \\ \omega_j \\ \omega_k \end{Bmatrix} \tag{13}$$

Where i, j, and k are the node numbers of the triangular element.

By applying the Galerkin method for a typical element as in Ref. [16] and [17], the energy equation (4) is given by

$$\iint_{\Omega^{(e)}} N_i \left[\left(\frac{\partial^2 t^{(e)}}{\partial x^2} + \frac{\partial^2 t^{(e)}}{\partial y^2} \right) - w^{(e)} \right] dxdy = 0 \tag{14}$$

Using the integration by parts and Green's theorem, equation (14) can be reduced to the following equation.

$$\iint_{\Omega^{(e)}} \left(\frac{\partial t^{(e)}}{\partial x} \frac{\partial N_i}{\partial x} + \frac{\partial t^{(e)}}{\partial y} \frac{\partial N_i}{\partial y} \right) dxdy$$

$$+\iint_{\Omega^{(e)}} N_i w^{(e)} dxdy$$

$$-\int_{S^{(e)}} \left(\frac{\partial t^{(e)}}{\partial x} n_x + \frac{\partial t^{(e)}}{\partial y} n_y\right) N_i d\Sigma^{(e)} = 0 \qquad (15)$$

Similary, the momentum equation (3) is given by

$$\iint_{\Omega^{(e)}} \left(\frac{\partial \omega^{(e)}}{\partial x} \frac{\partial N_i}{\partial x} + \frac{\partial w^{(e)}}{\partial y} \frac{\partial N_i}{\partial y}\right) dxdy$$

$$-R_a \iint_{\Omega^{(e)}} N_i \omega^{(e)} dxdy - L \iint_{\Omega^{(e)}} N_i dxdy$$

$$-\int_{S^{(e)}} \left(\frac{\partial w^{(e)}}{\partial x} n_x + \frac{\partial w^{(e)}}{\partial y} n_y\right) N_i d\Sigma^{(e)} = 0 \qquad (16)$$

Substtuting the flow boundary condition, $w=0$, and the thermal boundary condition, $\frac{\partial t}{\partial n} = \frac{\partial t}{\partial x} n_x + \frac{\partial t}{\partial y} n_y = 0.25G(x_c, y_c)$ as well as equation (10) and (11) into equation (15) and (16) gives

$$\iint_{\Omega^{(e)}} \left(\left[\frac{\partial N}{\partial x}\right]\{t^e\} \frac{\partial N_i}{\partial x} + \left[\frac{\partial N}{\partial y}\right]\{t^e\} \frac{\partial N_i}{\partial y}\right] dxdy$$

$$+\iint_{\Omega^{(e)}} [[N]\{w^e\} N_i] dxdy - C_1 \int_{S^{(e)}} N_i d\Sigma^{(e)} = 0$$

$$(17)$$

$$\iint_{\Omega^{(e)}} \left(\left[\frac{\partial N}{\partial x}\right]\{w^{(e)}\} \frac{\partial N_i}{\partial x}\right.$$

$$\left. + \left[\frac{\partial N}{\partial y}\right]\{\omega^e\} \frac{\partial N_i}{\partial y}\right] dxdy$$

$$-R_a \iint_{\Omega^{(e)}} [[N\{t^e\} N_i] dxdy - L \iint_{\Omega^{(e)}} N_i dxdy = 0$$

$$(18)$$

where $C_1 = 0.25G(x_c, y_c)$

We can write the equations (18) and (17) in matrix forms as

$$[F^e]\{w^e\} - R_a[M^e]\{t^e\} = L\{B^e\} \qquad (19.a)$$

$$[M^e]\{\omega^e\} + [F^e]\{t^e\} = C_1\{Q^e\} \qquad (19.b)$$

where

$$F_{ij} = \iint_{\Omega^{(e)}} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y}\right) dxdy$$

$$(20.a)$$

$$M_{ij} = \iint_{\Omega^{(e)}} N_i N_j dxdy \qquad (20.b)$$

$$B_i = \iint_{\Omega^{(e)}} N_i dxdy \qquad (20.c)$$

$$Q_i = \int_{S^{(e)}} N_i d\Sigma^{(e)} \qquad (20.d)$$

Summing up all the elements into a global matrix, equations (19) become a set of $2n$, linear, nonhomogeneous, algebraic equations for the $2n$ unknown variables, $w_n{}'(n=1, 2, \ldots n)$ and t_n $(n=1, 2, \ldots n)$.

After the nodal velocities and the temperaures are obtained, the bulk temperature and Nusselt number can be computed as follows.

The bulk temperature of the fully developed duct flow and of the constant axial heat flux is given by [3-12]

$$t_m = \frac{\iint \omega t dA}{\iint w dA} \qquad (21)$$

Using the nodal velocity and the temperature, equation (21) can be rewritten as

$$t_m = \frac{\sum_{e=1}^{n} \{\omega^e\}^T [M]\{t^e\}}{\sum_{e=1}^{n} \{B\}^T \{\omega^e\}} \qquad (22)$$

Where M and B are defined by equations (20.b) and (20.c). The Nusselt number, the object of present study, is given by [14]

$$Nu(x_c, y_c) = \frac{0.25G(x_c, y_c)}{\Delta t_w(x_c, y_c)}$$

where $\Delta t_w(x_c, y_c) = t_w(x_c, y_c) - t_m$ and x_c and y_c are the coordinates of the boundary.

4. Illustrative Examples

The problem of combined free and forced convection for the arbitrary shaped ducts under the boundary condition of (b) or (c) is solved using the finite element solution algorithms derived in previous section. The nodal values i, j, and k at both interior and boundary for each element as well as the value of Ra and L were used as input data to the computer program.

In order to compare the solutions obtained by finite element method with the analytical method, simple circular duct with the boundary condition of $\dot{q}''=\text{const}$ or $\dot{q}_0''\times(1+\cos\theta)$ is chosen for this purpose.

4.1. Circular Duct with the Boundary Condition of Uniform Peripheral Heat Flux.

The results of the computed Nusselt number are summerized in Table 1.

Table 1 The computed Nusselt number by FEM and exact solution.

Ra	FEM (89 nodes)	Exact solution [8]	Error (%)
1	4.387	4.36	0.62
100	4.4434	4.43	0.32
1000	4,9165	4.99	1.47
10000	7.8241	8.49	7.84

The values of the pressure gradient parameter L (Table 5 of reference [8]) used are shown in Table 2.

Table 2 The values of L.

Ra	L
1	32.06
100	37.69
1000	85.45
10000	413.91

4.2. Circular Duct with the Boundary Condition, $\dot{q}''=\dot{q}_0''(1+\cos\theta)$

The boundary condition of peripheral heat flux variation as shown in Fig.2, is studied.

For this case the boundary condition for flow is given by

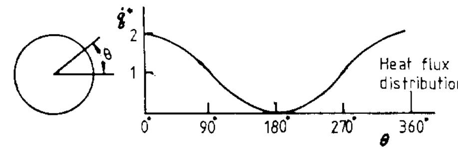

Fig. 2 Circular duct with heat flux distribution along the boundary.

$$w=0$$

and the thermal boundary condition is given by

$$\frac{\partial t}{\partial n}=0.25(1+\cos\theta)$$

where $\theta=\tan^{-1}\frac{x_c}{S}$, in which $S=\sqrt{x_c^2+y_c^2}$

For the pressure gradient parameter, same values are used as in section 4.1.

The computed Nusselt number and $\Delta t_w/\Delta t_{av}$ at Ra=1, 100, 1000, and 10000 are illustrated in Eig. 3, 4, 5 and 6.

It is obvious that the Nusselt number at Ra=1 agrees very closely with the Nusselt

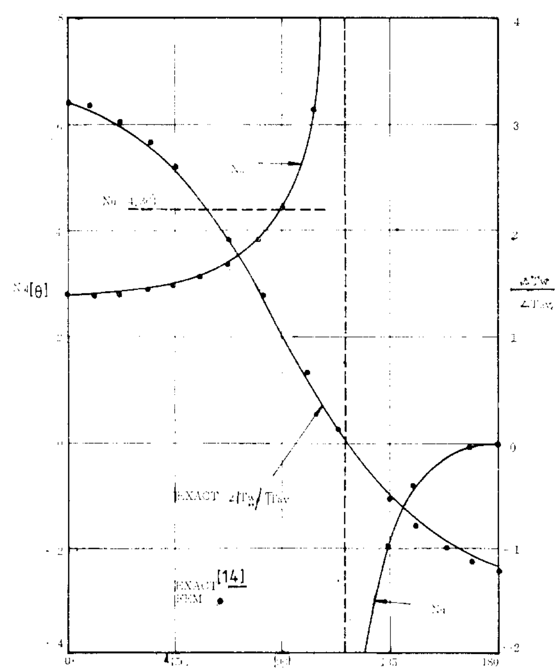

Fig. 3 Illustration of effect of circumferential heat flux variation in circular duct (Ra=1).

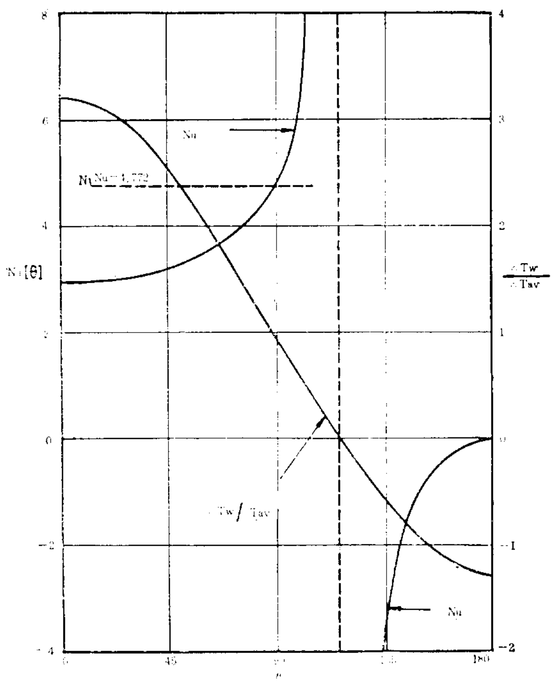

Fig. 4 Illustration of effect of circumferential heat flux variation in circular duct (Ra=100).

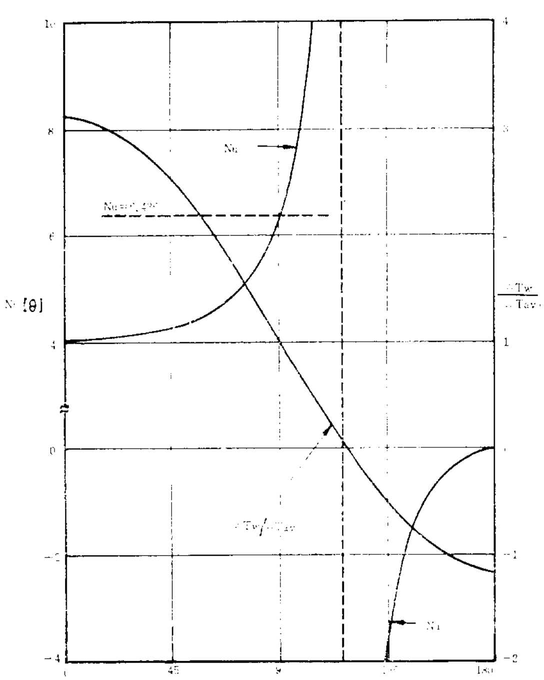

Fig. 5 Illustration of effect of circumferential heat flux variation in circular duct (Ra=1000).

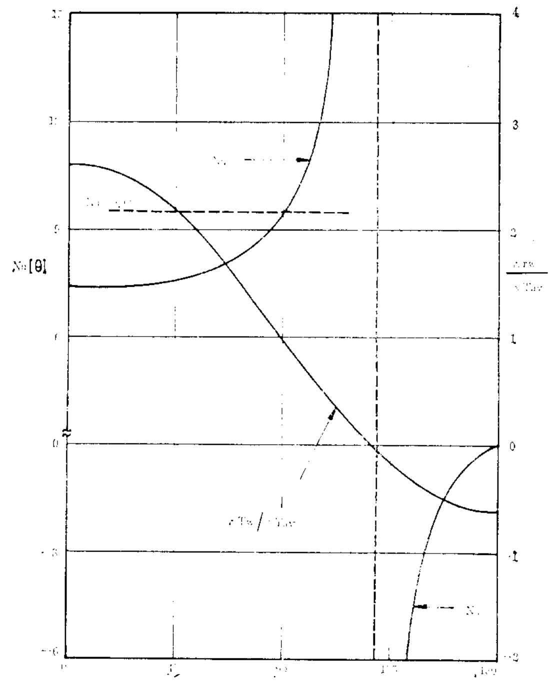

Fig. 6 Illustration of effect of circumferential heat flux variation in circular duct (Ra=10000).

number for the pure forced convection [14].

When the Ra value is high, exact solutions are not available, so comparison is not attempted.

5. Concluding Reamrks

The problem of combined free and forced convection with uniform peripheral heat flux and peripheral heat flux variation in arbitrary shaped ducts was studied using finite element solution algorithm.

As shown in the illustrative examples, the solutions obtained by finite element method in this study match very closely to the exact solution.

However there remains one problem, that is, the value of pressure gradient parameter L as an input datum should be available.

In case of boundary condition (a), this problem was solved in Ref [13]. To eliminate this disadvantage for the boundary condition (b) and (c), further study is to be continued.

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