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<**Original**>

Thermo-Mechanical Stress Analysis for Partial or Entire Crack Closure

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크랙의 部分 또는 完全닫힘에 관한 熱 및 機械的 應力解析

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抄 錄

Muskhelishvili의 複素數方法에 의해 一般的荷重 즉 熱 및機械的荷重을 받는 無限彈性體內에 空洞을 假定하고 그 周圍의 應力 및 變位를 誘導하였다. 線形크랙(line rack)이 部分的으로 또는 完全히 닫힐 臨界荷重條件과 그때의 應力세기係數(stress intensity factor)를 McClintork와 Walsh의 크랙닫힘에 관한 假定에 基礎를 두고 解釋學的으로 誘導 하였다.

Nomenclature

- $A=A_R+iA_I$: Complex constant
- A_R, A_I : Real and imaginary parts of complex constant A, respectively
- $a=R(1+m)$: Major axis of cavity
- $b=R(1-m)$: Minor axis of cavity
- A_n, B_n, C : Constants defined by thermal boundary conditions
- D_0 : Temperature constant
- $g(\xi)$: $\int[T(\rho,\theta)+iW(\rho,\theta)]dz$
- i : $(-1)^{1/2}$
- k : k_1-ik_2
- k_1, k_2 : Crack tip stress intensity factors for Mode I and II, respectively
- M : Material bulk modulus
- m : Parameter($0\leq m\leq 1$) which governs the shape of the cavity, i.e $m=1$ for a line crack and $m=0$ for a circular cavity
- P : Internal cavity pressure
- Q : Heat flow rate per unit length
- $R(\geq 0)$: Parameter which governs the size of the cavity
- $T(\rho,\theta)$: Temperature function
- T_0 : Temperature change on crack/cavity surface
- u_y : Displacement components in cartesian coordinates along y direction
- $w(\rho,\theta)$: Harmonic conjugate of $T(\rho,\theta)$
- x, y : Cartesian coordinates
- $z=x+iy$: Complex variable
- α : Coefficient of linear thermal expansion
- β : Elastic constant which takes values $\beta=(1+\nu)\alpha$ for plane strain and $\beta=\alpha$ for plane stress
- γ : Elastic constant which takes values $\gamma=3-4\nu$ for plane strain and $\gamma=(3-\nu)/(1+\nu)$ for plane stress.
- $\xi(=\rho e^{i\theta})$: Complex variable
- κ : Coefficient of thermal conductivity
- μ : Shear modulus
- μ_f : Material friction coefficient
- ν : Poisson's ratio
- ρ, θ : Curvilinear coordinates
- σ_c : Material constant as critical stress to close acrack at a certain location or entirely
- σ_n, σ_f : Crack closure contact normal and fric-

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	tional shear stresses, respectively
$\sigma_x, \sigma_y, \sigma_{xy}$	: Applied mechanical stress componentsin cartesian coordinates
σ_ρ, u_ρ	: Stress and displacement components normal to the curve ρ=constant, respectively
σ_θ, u_θ	: Stress and displacement components normal to the curve θ=constant, respectively
$\sigma_{\rho\theta}$	: Shear stress component in cuvilinear coordinates
$\varphi(\xi), \psi(\xi)$	: Kolosoff stress functions of complex variable
$\Phi(\xi)$	: $\varphi'(\xi)/\omega'(\xi)$
$\Psi(\xi)$	: $\psi'(\xi)/\omega'(\xi)$
$\omega(\xi)$	: Comformal mapping function
$Rl\{\ \}$, $Im\{\ \}$	: Real and imaginary parts of complex function { }, respectively
prime	: Derivative with respect to argument
bar	: Complex conjugate of complex function beneath the bar

1. Introducton

It has been demonstrated that under certain conditions, it is possible for opposite crack surfaces to meet at least once. By assuming that in compression, the Griffith crack closes and develops a friction force across the crack surfaces, McClintock and walsh [1] modified the Griffith failure theory. Burniston [2] showed that in a plate, the Griffith crack opened by a uniform tension at infinity is partially closed at the middle by equal and opposite point loads applied along the centerline of the crack. Tweed [3] proposed a method of handling two dimensional crack problems with a uniform tension at infinity and a symmetric system of body forces involving a partial closure at the middle. Burniston and Gurley [4] considered the cases of partial closures at the middle or at the ends of the Griffith crack under an internal parabolic pressure and a pair of concentrated forces located along the centerline of the crack. While the studies mentioned above are restricted to loading symmetrical with respect to the centerline of the crack, the author of the present work proposes a method of solution for partial or entire crack-closure at an arbitrary location of the crack under asymmetric thermomechanical applied stresses and internal pressure by using Muskhelishvili's cmplex variable method [5] and McClintock and Walsh's crack closure assumptions [1].

2. Stress and Displacement Expressions for Crack/Cavity Problems

The selected two-dimensional, thermo-mechanical elastic model with a non-interacting. isolated elliptic cavity is illustrated in Fig.1. This representation is a modification of McClintock and Walsh's crack/cavity closure model [1] which was used to derive the modified Griffith failure criterion. Under steady state temperature condition, the thermo-elastic stress and displacement fields in curvilinear coordinates (ρ, θ) (Fig.2) in terms of the Kolosoff stress functions, $\varphi(\xi)$ and $\psi(\xi)$, are written in the form [5, 6]

$$\sigma_\rho + \sigma_\theta = 4Rl\Phi(\xi) \tag{1}$$

$$\sigma_\theta - \sigma_\rho + 2i\sigma_{\rho\theta} = \frac{2\xi^2}{\rho^2\overline{\omega'(\xi)}}[\overline{\omega(\xi)}\Phi'(\xi) + \omega'(\xi)\Psi(\xi)] \tag{2}$$

$$2\mu|\omega'(\xi)|(u_\rho + iu_\theta) = \frac{\overline{\xi}\overline{\omega'(\xi)}}{\rho}\left[\gamma\varphi(\xi) - \frac{\omega(\xi)}{\overline{\omega'(\xi)}}\overline{\varphi'(\xi)} - \overline{\psi(\xi)} + 2\mu\beta g(\xi)\right] \tag{3}$$

where

$$\Phi(\xi) = \varphi'(\xi)/\omega'(\xi), \quad \Psi(\xi) = \psi'(\xi)/\omega'(\xi) \tag{4}$$

$$Z = \omega(\xi) = R\left(\xi + \frac{m}{\xi}\right) \quad (\xi = \rho e^{i\theta}) \tag{5}$$

and $g(\xi)=\int[T(\rho,\theta)+iW(\rho,\theta)]dZ$ (6)

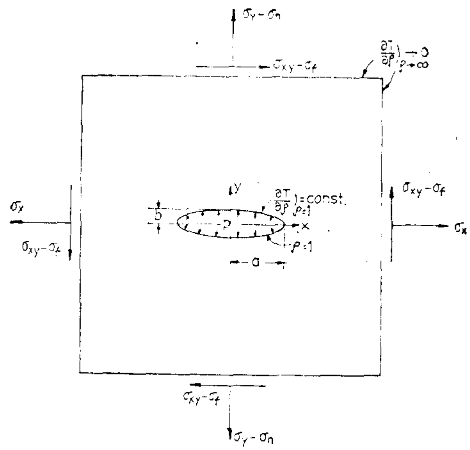

Fig. 1 Two demensional thermo-mechanical model.

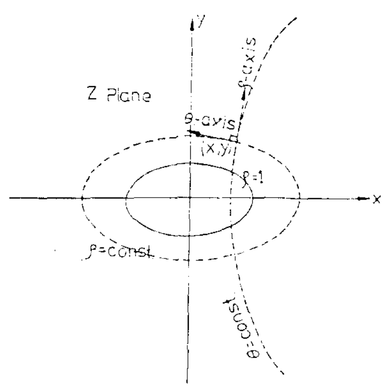

Fig. 2 Curvilinear coordinates (ρ,θ) of the point (x,y) of the Z plane.

By using Muskhelishvili's complex variable method, the stress functions for the model represented in Fig.1. can be shown in the form [5, 7, 8]

$$\varphi(\xi)=\frac{(\sigma_y-\sigma_n)R}{4}\left(\xi-\frac{2+m}{\xi}\right)+\frac{\sigma_x R}{4}\left(\xi+\frac{2-m}{\xi}\right)-\frac{PRm}{\xi}+i\frac{(\sigma_{xy}-\sigma_f)R}{\xi}+Aln\xi \quad (7)$$

$$\varphi(\xi)=-\frac{(\sigma_y-\sigma_n)}{2}R\left[-\xi-\frac{1}{m\xi}+\frac{(1+m^2)(1+m)\xi}{m(\xi^2-m)}\right]-\frac{\sigma_x R}{2}\left[\xi+\frac{1}{m\xi}-\frac{(1+m^2)(1-m)\xi}{m(\xi^2-m)}\right]-\frac{PR}{\xi}-\frac{PRm(1+m\xi^2)}{\xi(\xi^2-m)}+i(\sigma_{xy}-\sigma_f)R\left[\xi-\frac{1}{m\xi}+\frac{(1+m^2)\xi}{m(\xi^2-m)}\right]+\bar{A}ln\xi-\frac{A(1+m\xi^2)}{\xi^2-m} \quad (8)$$

where A is a complex constant related to thermal loading. Introducing equations (5), (7) and (8) into equetions (1) and (2), the stress fields in terms of arbitrary ρ and m values can be derived. The stress component, σ_θ, on $\rho=1$ is obtained in the form

$$\sigma_\theta=\frac{1}{1-2m\cos 2\theta+m^2}\{(\sigma_y-\sigma_n)(1+2\cos 2\theta-2m-m^2)+\sigma_x(1-2\cos 2\theta+2m-m^2)+P(-3m^2+2m\cos 2\theta+1)-4(\sigma_{xy}-\sigma_f)\sin 2\theta\}+\frac{1}{R(1-2m\cos 2\theta+m^2)}\cdot\{3[A_R(1-m)\cos\theta+(1+m)A_I\sin\theta]-\frac{(1+2m\cos 2\theta+m^2)}{(1-2m\cos 2\theta+m^2)}[A_R(1-m)\cos\theta+A_I(1+m)\sin\theta]+\frac{2(m^2+1)}{(1-2m\cos 2\theta+m^2)^2}\cdot[A_R\cos\theta(1+m-m^2-m^3+\cos^2\theta(4m^2-4m))+A_I\sin\theta(1-m-m^2+m^3+\sin^2\theta(4m+4m^2))]\} \quad (9)$$

For the determination of the unknown function $g(\xi)$ in equation (3), the following temperature boundary conditions are assumed : (i) steady state temperature condition (ii) temperature is assumed to be continuous across $\rho=1$ (iii) the temperature gradient with respect to ρ, $\partial T/\partial\rho$, decreases abruptly by a prescribed function $G(\theta)$ as ρ increases from $\rho=1-\epsilon$ to $1+\epsilon$ with small ϵ (iv) $\partial T/\partial\rho$ vanishes at $\rho\to\infty$.

Using the above assumptions for temperature distribution, the solution for the temperature becomes

$$T=T_0-C\ln\rho+\sum_{n=1}^{\infty}\rho^{-n}(A_n\cos n\theta+B_n\cos n\theta) \quad \text{in } 1\leq\rho<\infty \quad (10)$$

where

$$A_n=\frac{1}{2\pi n}\int_0^{2\pi}G(\theta)\cos n\theta\,d\theta$$

$$B_n=\frac{1}{2\pi n}\int_0^{2\pi}G(\theta)\sin n\theta\,d\theta \qquad (11)$$

$$C=\frac{1}{2\pi}\int_0^{2\pi}G(\theta)d\theta \quad (n=1,2,3\ldots)$$

and T_0 is a constant.

The quantity $W(\rho,\theta)$ in equation (6), obtained from the Cauchy-Riemann relations, is give by

$$W=D_0-C\theta-\sum_{n=1}^{\infty}\frac{1}{\rho^n}(A_n\sin n\theta-B_n\cos n\theta)$$

$$\text{in } 1\leq\rho<\infty \qquad (12)$$

where D_0 is a constant.

Introducing equations (5), (6), (7), (8), (10) and (12) into equation (3) and considering $u_\rho+iu_\theta$ as a single valued and continous function, we can obtain lengthy expressions for displacements. For the case $\rho=1$ and $G(\theta)=$ constant, along with the definition of heat flow rate per unit cavity length as

$$Q_{\rho=1}=2\pi\cdot\kappa\left(\frac{\partial T}{\partial\rho}\right)_{\rho=1} \qquad (13)$$

and with the symmetric condition of the thermal displacement component of u_y, the displacement expression u_y is obtained in the form

$$\begin{aligned}
&2\mu(1-2m\cos 2\theta+m^2)^{1/2}u_\rho\\
&=\frac{(\sigma_y-\sigma_n)R}{4}\{\gamma[1-(2+2m)\cos 2\theta\\
&\quad+(2+m)m]+\frac{1}{(1-2m\cos 2\theta+m^2)}\cdot\\
&\quad[1+4m+4m^2+2m^3+m^4+(-4m^3-6m^2\\
&\quad-4m-2)\cos 2\theta+(2m^2+2m)\cos 4\theta]\}\\
&\quad+\frac{\sigma_xR}{4}\{\gamma[1+(2-2m)\cos 2\theta-(2-m)m]\\
&\quad+\frac{1}{(1-2m\cos 2\theta+m^2)}[1-4m+4m^2\\
&\quad-2m^3+m^4+(-4m^3+6m^2-4m+2)\cos 2\theta\\
&\quad+(2m^2-2m)\cos 4\theta]\}-PRm\{\gamma(\cos 2\theta\\
&\quad-m)+\frac{\cos2\theta(1-m^2)-m\cos 4\theta+m^3}{1-2m\cos 2\theta+m^2}\\
&\quad-\frac{1+m^2}{m}\Big\}+(\sigma_{xy}-\sigma_f)R\sin 2\theta(1+\gamma)\\
&\quad+2\mu\beta R\{(1-m)\cos^2\theta[T_0(1+m)\\
&\quad-\frac{Q_{\rho=1}}{2\pi\lambda}(1-m)\Big]-(1+m)\sin^2\theta\cdot\\
&\quad\left[\frac{Q_{\rho=1}}{2\pi\lambda}(1+m)-T_0(1-m)\right]\Big\}
\end{aligned} \qquad (14)$$

where

$$A=A_R+iA_I=-\frac{\mu\beta R(1+m)Q_{\rho=1}}{\pi\lambda(1+\gamma)} \qquad (15)$$

and T_0 is the cavity surface temperature change.

3. Critical Loading Conditions for Crack Closure

Under specified loading conditions, it is possible for the crack/cavity to close at certain locations or to close entirely. For partial or entire crack/cavity closure, the following conditions should be satisfied at the crack/cavity surface, $\rho=1$.

(i) for crack/cavity closure at its ends

$$u_y=0 \text{ and } \frac{\partial u_y}{\partial x}=0 \quad (\theta=0,\pi) \qquad (16)$$

(ii) for crack/cavity closure at a location other than its ends

$$u_y=-R(1-m)\sin\theta \quad (\theta\neq0,\pi) \qquad (17)$$

(iii) for entire crack/cavity closure

$$u_y=-R(1-m)\sin\theta \quad (\text{all } \theta) \qquad (18)$$

Although analysis for a general value of m has been conducted, the specific case of a line crack, $m=1$, is reported here.

By considering relations between curvilinear and cartesian coordinates, inserting equation(14) into equations (16), (17) and (18) and defining critical load as the load at the instant prior to crack-closure at a certain location or entire closure, the following critical loading conditions for closure of a line crack are obtained.

(i) critical loading condition for crack-closure at its tips $(\theta=0,\pi)$

$$\sigma_{xy}=0 \text{ and } \frac{1+\gamma}{2\mu}(\sigma_y+P)=\frac{\beta Q_{\rho=1}}{\pi\kappa} \qquad (19)$$

(ii) critical loading conditions for crack-closure at a location (θ) other than its ($\theta \neq 0, \pi$)

$$\left\{\frac{1+\gamma}{2\mu}(\sigma_y+P)-\frac{\beta Q_{\rho=1}}{\pi\kappa}\right\}\sin\theta + \frac{1+\gamma}{2\mu}\sigma_{xy}\cos\theta=0 \quad (20)$$

(iii) critical loading conditions for entire crack-closure (all θ)

There are two possibilities for entire crack closure. The crack can be resulted in entire closure at certain loads by gradually spreading the local closed area following local closure at a location. The critical loads in this case are governed by

$$\sigma_{xy}=\sigma_f \text{ and } \frac{1+\gamma}{2\mu}(\sigma_y-\sigma_n+P)=\frac{\beta Q_{\rho=1}}{\mu\kappa} \quad (21)$$

If the crack closes entirely without prior local closure, the critical loading conditions can be shown to be identical to equation (19).

By comparing the critical loading conditions for each case, it can be seen that the applied shear stress, σ_{xy}, plays an important role in defining the mechanisms for crack closure. When applied shear stress is equal to zero, all the critical loading conditions for the different cases become same and are specified by equation (19). Therefore, in this case, the crack will close entirely at the loads given by equation (19) without experiencing the prior local closure. When applied shear stress is non-zero, the crack will not close at its tips but locally or entirely following local closure. Consequently, since the case that the crack begins to close at its tips will not occur, it is not discussed further. McClintock and Walsh's assumptions for crack closure [1] are extended here for thermo-mechanical loads. The modified McClintock and Walsh thermo-mechanical crack closing concept may be written in the form

$$\sigma_n=0 \text{ for either } \sigma_y \geq 3\alpha MT_0 \text{ or } 0 \geq \sigma_y-3\alpha MT_0 \geq \sigma_c \quad (22)$$

$$\sigma_n=\sigma_y-3\alpha MT_0-\sigma_c \text{ for } \sigma_y-3\alpha MT_0<\sigma_c<0 \quad (23)$$

$$\sigma_f=-\mu_f\sigma_n \text{ for } \sigma_{xy}\geq\sigma_f \quad (24)$$

In the absence of thermal loading, equations (22), (23) and (24) reduce to McClintock and Walsh's equations. In order to determine the normal contact stress, σ_n, the critical stress, σ_c, is obtained as follows:

(i) for crack-closure at a location (θ) other than its tips ($\theta \neq 0, \pi$)

By the definition of the critical load, σ_n becomes zero in this case. Therefore

$$\sigma_y=3\alpha MT_0+\sigma_c \quad (25)$$

Introducing equation (25) into equation (20), we obtain

$$\sigma_c=-P-3\alpha MT_0+2\mu\beta Q_{\rho=1}/\{\pi\kappa(1+\gamma)\} -\sigma_{xy}\cot\theta \quad (26)$$

(ii) for entire crack-closure resulting from contact spreading due to prior local closure

In this case σ_n does not vanish. Therefore, equation (23) is introduced into equation (21) to yield

$$\sigma_c=-P-3\alpha MT_0+2\mu\beta Q_{\rho=1}/[\pi\kappa(1+\gamma)] \quad (27)$$

(iii) for entire crack-closure without prior local closure

In this case, σ_c can be shown to be identical with equation (27)

4. Stress Intensity Factors for Partially or Entirely Closed Crack

The pertinent definition of the stress intensity fact or which exist only for the case of a "sharp" crack, *i.e.* $m=1$.

$$k=k_1+ik_2=2\sqrt{2}\lim_{\xi\to\xi_1}\{\omega(\xi)-\omega(\xi_1)\}^{1/2}\Phi(\xi) \quad (28)$$

Using equations (4), (5) and (7), we obtain from equation (28)

$$k_1=\left(\sigma_y-\sigma_n+P+\frac{2A_R}{a}\right)\sqrt{a} \quad (29)$$

$$k_2=\left(\sigma_{xy}-\sigma_f-\frac{2A_I}{a}\right)\sqrt{a} \quad (30)$$

By introducing equation (15) and the corresponding values of σ_n and σ_f obtained by equations (23) and (24) along with the pertinent value of σ_c into equations (29) and (30), we obtain the following stress intensity factors for the partially or entirely closed crack

(i) for the case that the crack is closed at a location other than its tips $(\theta \neq 0, \pi)$

$$k_1 = -\sigma_{xy} \cot\theta \sqrt{a} \quad (31)$$

$$k_2 = \{\sigma_{xy} + \mu_f [\sigma_y + P - 2\mu\beta Q_{\rho=1} / (\pi\kappa(1+\gamma)) + \sigma_{xy} \cot\theta]\} \sqrt{a} \quad (32)$$

(ii) for the case that the crack is closed entirely due to spreading of the local closure site

$$k_1 = 0 \quad (33)$$

$$k_2 = 0 \quad (34)$$

(iii) for the case that the crack is closed entirely without prior local closure

$$k_1 = 0 \quad (35)$$

$$k_2 = \mu_f [\sigma_y + P - 2\mu\beta Q_{\rho=1} / \{\pi\kappa(1+\gamma)\}] \sqrt{a} \quad (36)$$

It is noted that although the crack closes entirely, the stress intensity factor k_2 in this case is not zero, since applied shear stress σ_{xy} is zero but contact frictional shear stress σ_f is not. An interesting observation is that for the partially closed crack, the stress intensity factor k_1 (equation (31)) depends σ_{xy} and the location (θ) of a local crack closure rather than σ_y, P and thermal loading.

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