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Mathematical Models for Hit Probabilitles using Small-arms against Fast Low Flying Aircraft

Park, Chan Tae *

ABSTRACT

Mathematical models for hit probabilities of small arms are developed in order to estimate the expected hits on an aircraft for certain altitudes and air speeds. A model for the firing lead angle is developed for cases when the distribution of hits is normal and the firing angle is from 20 degrees to 160 degrees. probabilities of hit for single and multiple shots at various altitudes are calculated. Tables are given showing the probability of hits and kill for targets flying at high speed above 500 feet from ground level.

* Office of OR/SA, ROKAF College

I . INTRODUCTION

Seoul, capital of South Korea, is located very close to the D.M.Z. Seoul has almost 20 percent of the population of South Korea, many industrial facilities, and important military installations. Accordingly, air defense for Seoul is a matter of some importance to the government of the Republic of Korea. Attacks from medium and high altitude hostile aircraft can be countered with missiles and friendly aircraft, but air defense is very difficult when attacking circraft fly fast at low altitude. In the Viet Nam War, many low-flying aircraft were downed by small-arms fire.

The objective of Small arms fire is to:

1. Kill or damage hostile aircraft,
2. Reduce the efficiency of the hostile pilot,
3. Increase boming accuracy errors, and
4. Reduce the opponent's fighting spirit.

There are two techniques for employing small-arms fire against aircraft. In one technique, the gunner estimates the target's current line of flight and continuously adjusts his aim point to provide the approximate lead while firing. The other technique uses an arbitrarily selected lead angle so that the aircraft flies through a stream of bullets as it flies over the gunner. Both techniques use barrage fire to be effective: this method requires the firing of several weapons simultaneously at a common area in space in advance of the aircraft's flying path.

It is the objective of this thesis to explore these tactics by constructing a mathematical model for the probability of hitting fast low-flying aircraft with small-arms fire, and to find when the gun mores with the aircraft.

A basic model for the probability of hit for small-arms fire against fast low-flying aircraft will be developed in Section II. Only flight paths at conetant altitude with the approaching will be considered.

The general theoretical formations to be developed involve a density function and some basic geometry which will be used for calculating hit probabilities. Relative velocities between bullet and aircraft will be considered for the fixed gun case.

Section IV presents conclusions and recommendations from this study. In appendix A, we will discuss aircraft kill probabilities with reference to vulnerable (lethal areas) area of the target, when the gun is fixed at a certain aiming angle.

Ⅱ. DEVELOPMENT OF MODEL OF HIT PROBABILITY

In this chapter we will consider the formulation of the standard deviation (σ) of the bullet impact point, the formulation of the aircraft target area, and the model of hit probabilities. We will use the projected target area as the aircraft moves forward, since the angle between the gun and aircraft is changing.

In order to develop a hit probability model, we have to be concerned with the various factors shown in Figure 5 (Factors Affecting

Hit Probabilities).

A . FORMULATION OF STANDARD DEVIATION (σ)

We will look first at the gun dispersion angle, as shown in Figure 6, (Standard Deviation Geometry) in order to relate this to the standard deviation of a normal distribution.

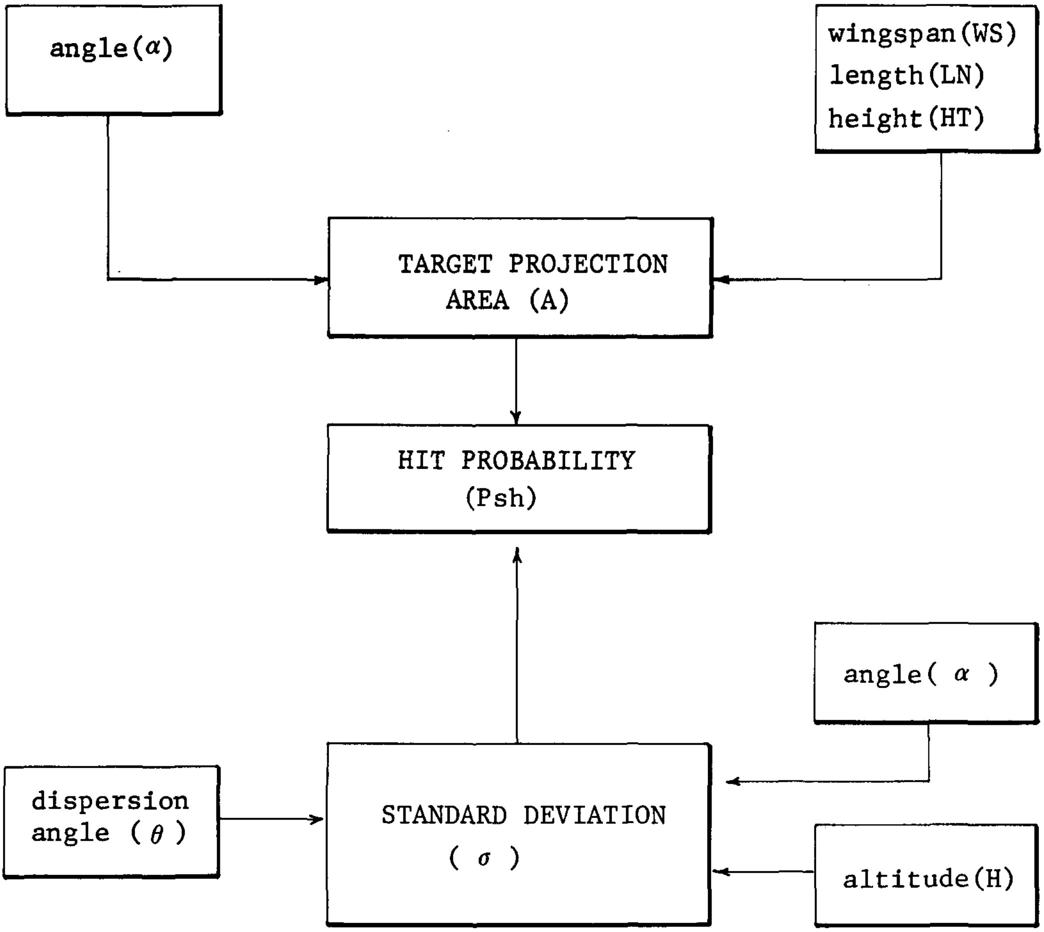

Figure 5 . Factor affecting hit probability(Psh)

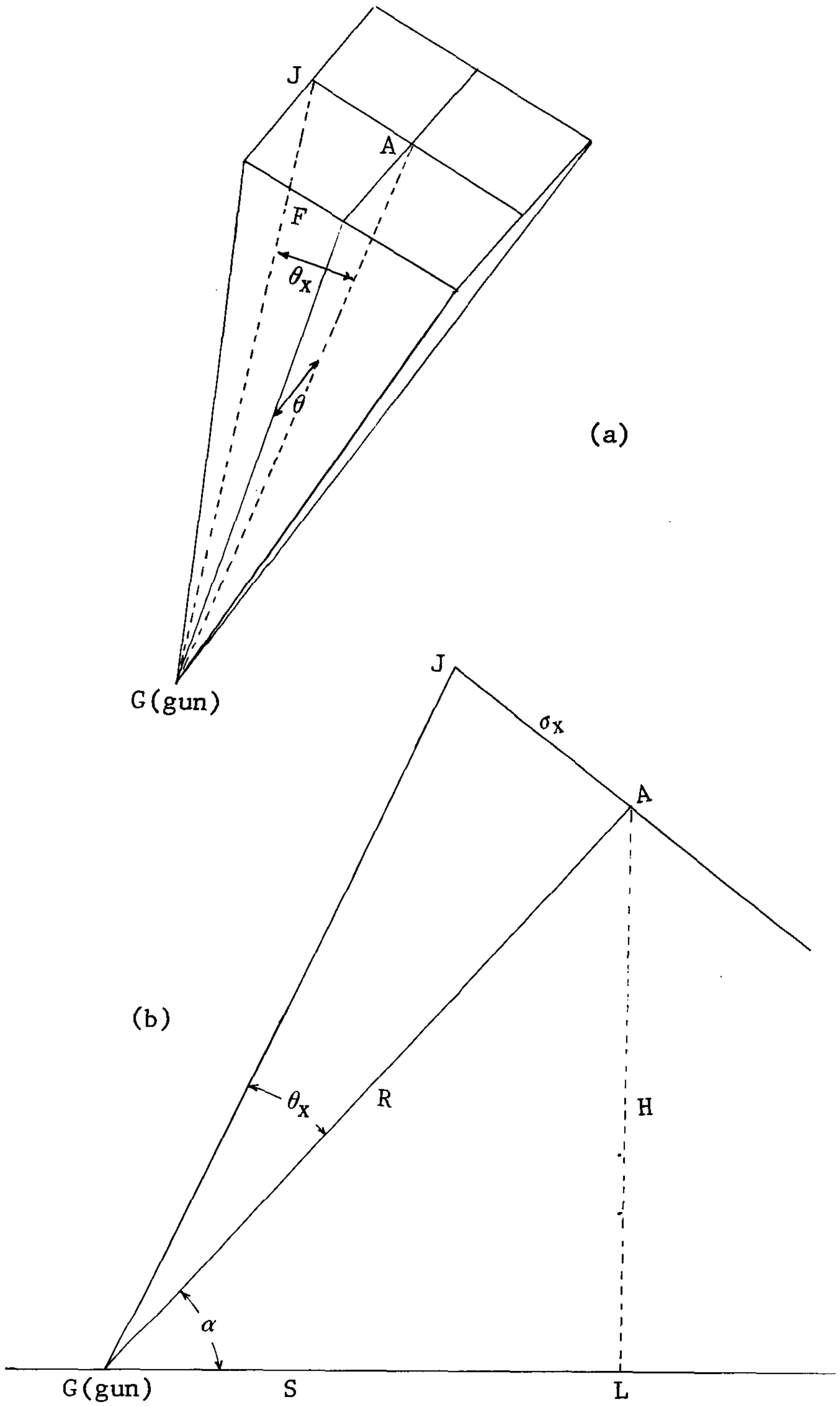

Figure 6 . Standard deviation (σ) geometry

Let θ_X be the dispersion azimuth angle and θ_Y the dispersion elevation angle of the bullet (known for a certain type of gun or bullet) from the triangle GAJ, as shown in Figure 6(a).

Let us assume that the dispersion angle (θ) is equal to θx , θy ($\theta = \theta_x = \theta_y$), and that this dispersion is normally distributed ($\mu = 0$, σ^2) and independent for each bullet, ($\sigma_x = \sigma_y = \sigma$). Then

$$\tan \theta_X = \frac{\overline{JA}}{\overline{GA}} \quad , \tag{27}$$

$$\sigma_x = \overline{GA} \cdot \tan \theta_x \quad , \quad \text{and} \tag{28}$$

$$\sin \alpha = \overline{LA} \;/\; \overline{GA} \quad \text{where} \quad \overline{LA} = H \; . \tag{29}$$

From equations (27), (28) and (29)

$$\sigma_x = (H \cdot \tan \theta x) \;/\; \sin \alpha \, , \tag{30}$$

and from the triangle GFA in Figure 6(a),

since $\sigma_x = \sigma_y = \sigma$, $\theta x = \theta y = \theta$, therefore the variance of the dispersion is

$$\sigma^2 = \frac{H^2 \cdot \tan^2 \theta}{\sin^2 \alpha} \tag{31}$$

In equation (31) we have shown the variance of dispersion is related to elevation of the target, the aiming angle and the dispersion angle. We are now ready to apply the distribution of round impacts to a target, whose size and shape is the subject of the next section.

B . FORMULATION OF TARGET AREA

In this section, we will use the projected target area as the angle between the gun and the aircraft is changed. It is very difficult to know the aircraft's projected area. We will use the basic aircraft dimensions (length, wingspan, height) since these dimensions are known. We will compute the bottom projected area and the forward projected area, and ignore the side projected area because we assume that the aircraft will be moving directly overhead.

We assume that the aircraft looks like a small box as shown in Figure 7.

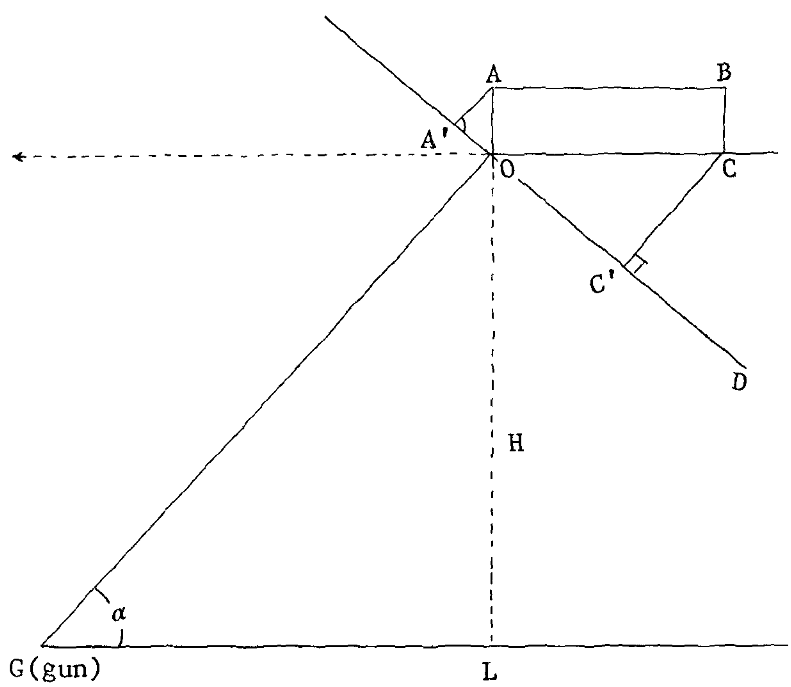

Figure 7 . Geometry of Projected Length

The impact point of the bullet on the aircraft may be considered as normally distributed on a line perpendicular to the ballistic trajectory, as shown in Figure 7.

In Figure 7:

$\overline{OC}$: length (LN)

$\overline{AO}$: height (Ht)

α : Angle of Gun - Target Line

G : Gun position

Let us assume that the aircraft looks like a small diamond-shaped box when we see the bottom area and small box when we see the forward and side as shown in Figure 8.

Thus,

Total Projected area (A)=forward Projected area (A_1) + bottom projected area (A_2).

The projection of height (HT) to the line perpendicular gun-target (Figure 7) is

$$\overline{OC'} = \overline{OA} \cdot \cos\alpha$$

$$= HT \cdot \cos\alpha .$$

let $\overline{OA'} = HT'$ or

$$\overline{HT'} = HT \cdot \cos\alpha \qquad (32)$$

The projection of length (LN) to the line is;

$$\overline{OC'} = \overline{OC} \cdot \sin\alpha$$

$$= LN \cdot \sin\alpha .$$

Let $\overline{OC'} = LN'$

where $LN' = LN \cdot \sin\alpha$

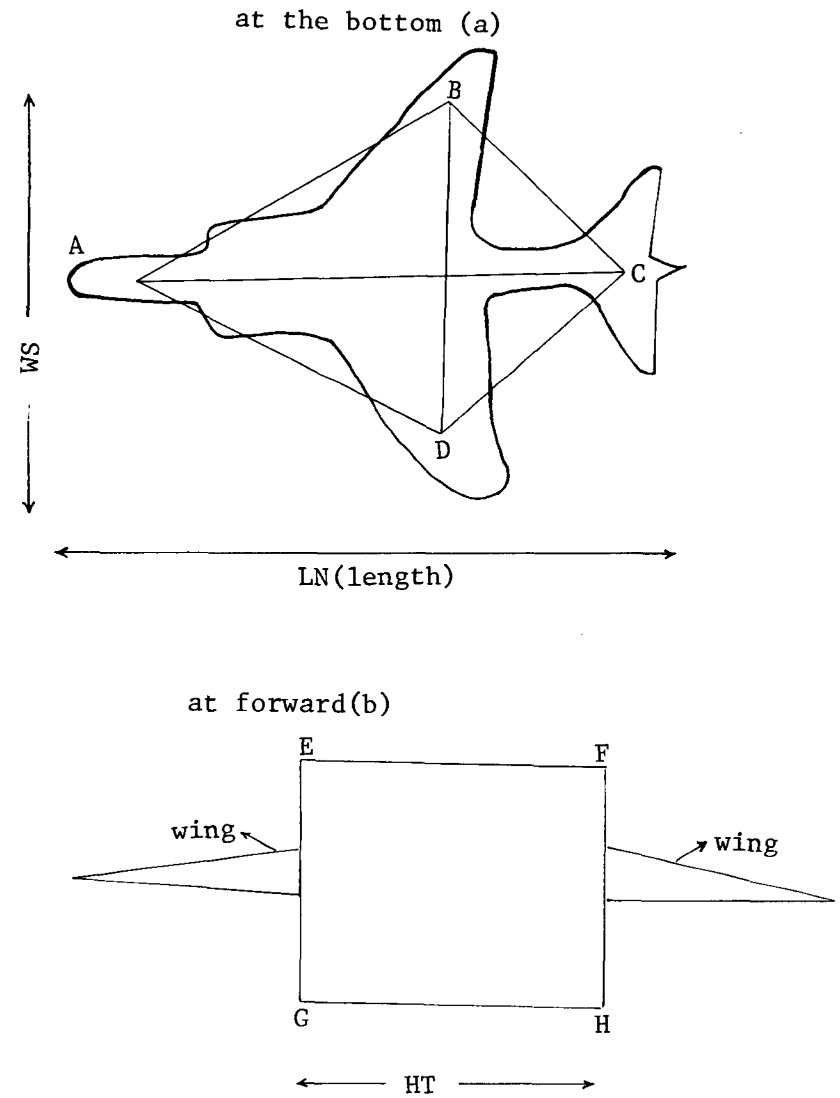

Figure 8 . Area of the Aircraft

Let us think about the projected forward area (A_1'), as shown in Figure 9(b) and ignore the front wing area since it looks like a very small area. Now, projected forward

area (A_1') = $\overline{EG}$ x $\overline{EF}$

= projected height (HT) x Wingspan (WS) x b

where b is ratio of fuselage length to wing span:

b = $\overline{EF}$ / Wing Span (WS) as shown in Table 4.

Therefore, the projected forward area (A_1') is;

$$A_1' = HT' \cdot b \cdot WS$$

$$= b \cdot ws \cdot HT \cdot \cos\beta \quad , \qquad (33)$$

where $0 < \beta < 90$ degree.

Let us think now about the projected bottom area (A_2')

where A_2 = Actual bottom area (A_2) $\times$ $\sin\alpha$, (34)

as shown in Table 4.

Total projected area is

$$A' = A_1' + A_2'$$

$$= b \cdot WS \cdot HT \cos \quad + A_2 \sin\alpha . \qquad (35)$$

Actual bottom area is very difficult to calculate. Accordingly, we will consider the diamond shape (bottom area) as shown in Figure 8. The diamond area is;

$$A_2 = 1/2 \ . \ \overline{AC} \cdot \overline{BD}$$

Let $\overline{AC}$ = c . WS

and $\overline{BD}$ = k . LN ,

where C, k is a factor for changing wing span and length.

(feet)

Name of Aircraft	Wing Span (WS)	Length (LN)	Height (HT)	Ratio (b)	Bottom Area (A1)	d
MIG-21	23.46	51.71	6.93	.27	430.0	.71
MIG-23	46.7	55.0	6.19	.264	836.0	.643
MIG-25	47.75	73.15	8.6	.2591	1188.7	.68
SU-7	29.36	57.0	6.9	.28	568.99	.68
MK-3	30.99	36.86	5.335	.264	388.4	.68
F-14A	64.05	62.34	8.06	.265	1356.0	.68
F-16A	31.0	47.04	6.437	.27	490.0	.67
F-18A	40.4	56.0	6.4	.261	768.0	.68
F-15A	42.8	63.9	7.13	.27	984.0	.72
F-4E	38.625	63.0	7.875	.262	808.0	.66
A-10A	57.6	53.4	7.31	.258	986.0	.65

Table 4 . Combat Aircraft Dimensions

Then, A_2' (projection bottom area) $= 1/2 \cdot c \cdot k \cdot WS \cdot LN'$

Let $c \cdot k = d$, as shown in Table 4. Then

$$\begin{aligned} A_2' &= 1/2 \cdot d \cdot WS \cdot LN' \\ &= 1/2 \cdot d \cdot WS \cdot LN \cdot \sin\alpha \\ &= \text{actual projection area } (A_2') . \end{aligned} \tag{36}$$

From equation (33) through (36) total projected area is

$$\begin{aligned} A' &= A_1' + A_2' \\ &= b \cdot WS \cdot HT \cos\alpha + 1/2 \cdot d \cdot WS \cdot LN \cdot \sin\alpha \end{aligned} \tag{36-1}$$

This projected area is related to basic aircraft dimensions (length, wingspan, height) and the aiming angle, will be used as target size for the hit probability in the next section.

C. SINGLE-SHOT HIT PROBABILITIES

In this section, we will consider the probability of hitting the projected area for the two dimensional case. (In the one-dimensional case we consider the aircraft as a rectangular target.) The impact point on the target is normally distributed on the line perpendicular to the ballistic trajectory). In the two-dimensional case the impact point of the bullet is on a plance which is perpendicular to the bullet trajectory.

Let the impact point of the bullet on a plane which is perpendicular to the bullet trajectory and which coincides with the center of the aircraft be presented by random variables X and Y.
It is assumed that X, Y are independently and normally distributed random variables with mean (μ) = 0, and variance ($\sigma = \sigma_x = \sigma_y$)

The density functions of X and Y are:

$$f_x(x) = (1/\sqrt{2\pi}\sigma_x)\ e^{-\frac{x^2}{2\sigma_x^2}} \tag{37}$$

and

$$f_y(y) = (1/\sqrt{2\pi}\sigma_y)\ e^{-\frac{y^2}{2\sigma_y^2}} \tag{38}$$

The joint density function is

$$f_{x,y}(x,y) = (e^{-\frac{x^2}{2\sigma_x^2}} / \sqrt{2\pi}\cdot\sigma_x) \cdot (e^{\frac{y^2}{2\alpha_y^2}} / \sqrt{2\pi}\ \sigma_y)$$

$$= (1/2\pi\sigma^2) \cdot e^{-\frac{x^2+y^2}{2\sigma^2}} , \tag{39}$$

where $\sigma_x = \sigma_y = \sigma$.

The probability of hit in an area A is

Psh(single shot hit probability)

$$= \iint_A 1/2\pi\sigma^2 \cdot e^{-\frac{x^2+y^2}{2\sigma^2}} dx \cdot dy . \tag{40}$$

Let us assume that the target is circular with radius R, so that

$$A_1 + A_2 = \pi R^2, \tag{41}$$

and $R^2 = (A_1 + A_2) / \pi$.

Transforming to polar coordinates, we have

$$\left.\begin{aligned} X &= r \cos\alpha , \\ Y &= r \sin\alpha \\ X^2 + Y^2 &= r^2 , \end{aligned}\right\} \tag{42}$$

and $dx.dy = r.dr.d\theta$

From equations (40) through (42)

$$Psh = \int_0^R e^{-r^2/2\sigma^2} \cdot r/\sigma^2 . dr\, d\theta . \tag{43}$$

Let $U = r^2 / 2\sigma^2$, $du = -r/\sigma^2 . dr$,

$r2 = 2\sigma^2 u = (A_1 + A_2)/\pi$,

and

$$U = (A_1 + A_2)/ 2\pi\sigma^2 . \tag{44}$$

From equations (43) and (44),

$$Psh = e^{-u}\, du \ ,$$

$$= 1 - e^{-u} \ .$$

From equation (44), then, we have

$$\boxed{Psh = 1 - e^{-(A_1 + A_2)/2\pi\sigma^2} \ ,} \qquad (45)$$

where A_1 is the projected forward area,

A_2 is the projected bottom area, and

σ^2 is variance for normal distribution.

This single-shot hit probability model will be used for the multiple-shot hit probability in the next section.

D. MULTIPLE-SHOT PROBABILITIES

In this section, we develop the hit probabilities for the multiple-shot case. We will discuss the relative speed between bullets and the aircraft, possible intercept of the bullet and the aircraft, and projected area as shown in Figure 9, (The Factors Affecting Multiple-Hit Probabilities).

First we will consider that the gun is fixed at a certain given aiming angle, and second, we will discuss the gun moving continously in a certain lead angle with the moving aircraft.

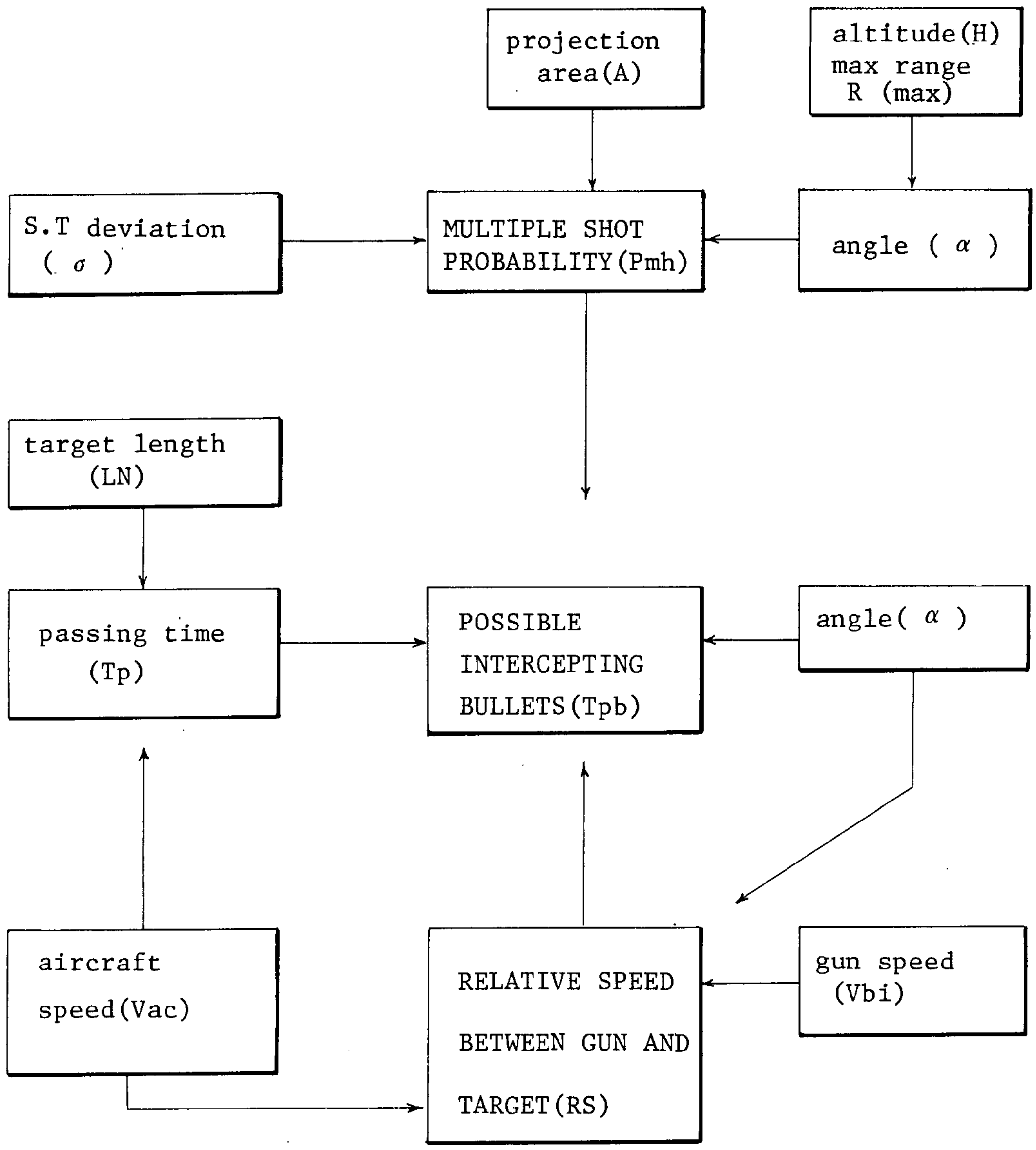

Figure 9 . Factors affecting multiple hit probability (Pmh) for gun fixed.

1 . The case when the gun is fixed with a certain given aiming angle .

In this section, we will discuss the model of multiple hit probabilities when the gun is fixed at a certain given aiming angle, as shown in Figure 10. Sometimes one person will fire continuously until aircraft passes the area. Figure 10, illustrates that the aircraft passes the interval which is determined by the aircraft length and projected height. It will take some time until the aircraft passes the interval.

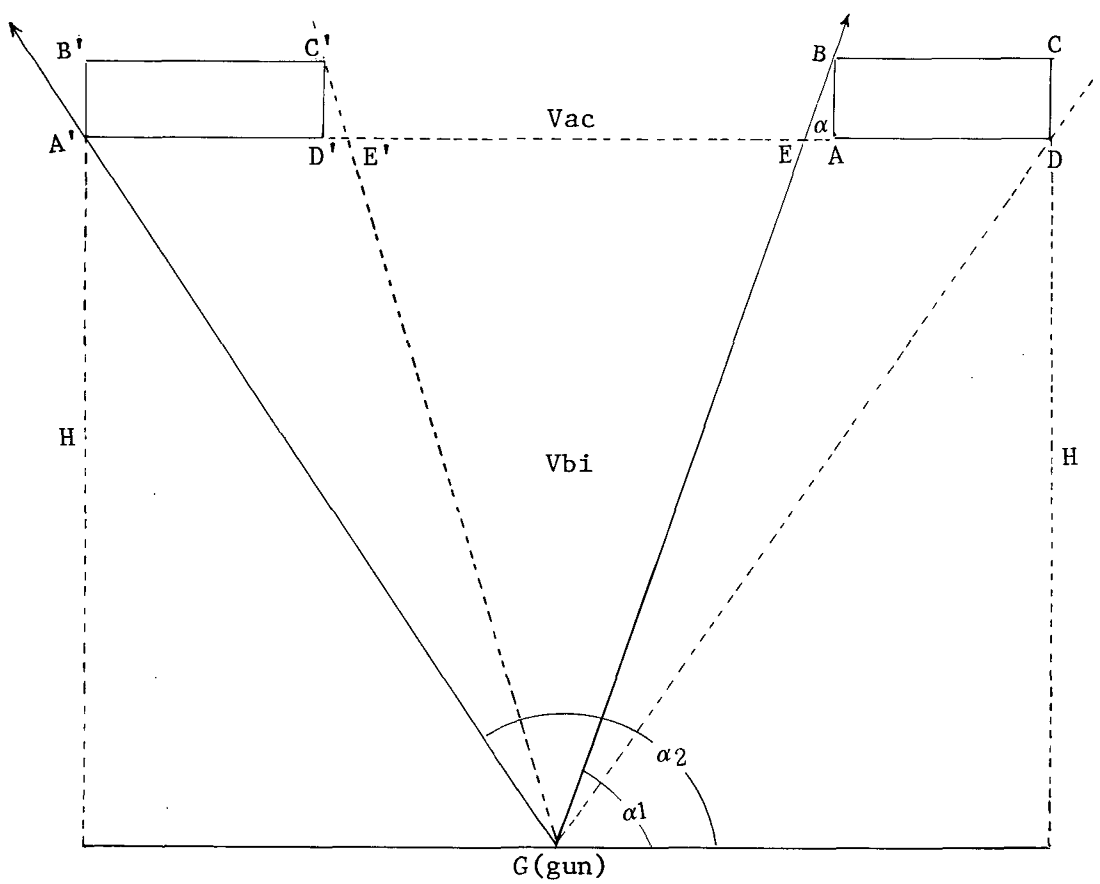

Figure 10 : Gun Fixed Geometry

In Figure 10:

α_1, α_2 : Fixed Angle

G : Gun Position

H : Altitude (constant)

Vbi : Muzzle Velocity

Vac : Aircraft Speed

Let us assume the gun will be fixed a certain aiming angle, since the aircraft is too fast to permit us to aim and follow the target We need to know how many bullets (Tpb) intercept the aircraft during aircraft passing time (Tp), which depends on the aircraft length (LN) and aircraft speed (VAC), as shown in Figure 10.

(1) Let projected possible target length (LN) be

$$LN = \overline{EA} + \overline{AD} ,$$

$$= \frac{\overline{AB}}{\tan\alpha} + \overline{AD} ,$$

$$= \frac{HT}{\tan\alpha} + LN, \text{ (ft)} \quad (46)$$

and aircraft passing time (Tp) be

$$Tp = \frac{LN'}{BAC} \quad (47)$$

(2) Let relative speed between bullet and aircraft be RS, as shown in Figure 11, so that

$$RS = VBI + VAC \cos\alpha , \quad (48)$$

and

$$\cos(\pi - \alpha) = -\cos\alpha .$$

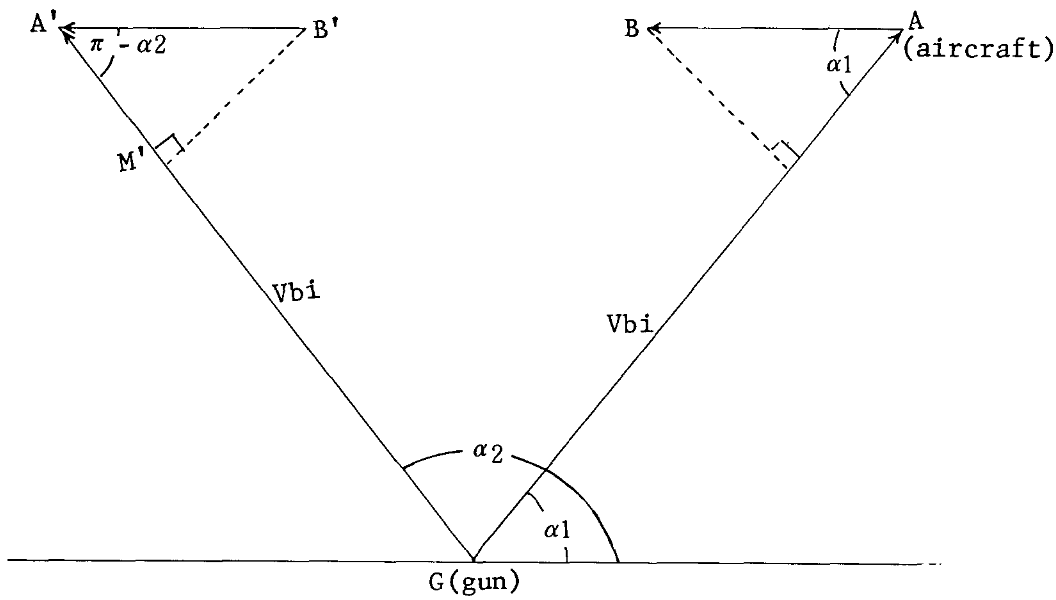

Figure 11 : Relative Speed

(3) Let interval between each bullet be IRD, then

$$IRD = \frac{\text{Bullet Speed}}{\text{Round per Sec}} = \frac{VBI}{RD} . \tag{49}$$

From equations (46), (47), (48), and (49), the total possible number of passing bullets are Tpb during the time period (0, Tp). Since

$$E(hits) = RS/IRD \ . \ dt ,$$

$$= RS/IRD \ . \ Tp , \tag{50}$$

then,

$$Tp = \frac{(Vbi + Vac.\cos\alpha)}{(\ Vbi/RD\ )} \ . \ \frac{(HT/\tan\alpha + LN)}{(VAC)} ,$$

and we have

$$Tp = \frac{RD\ (VBI + VAC\ \cos\alpha)\ (HT + LN\ \tan\alpha)}{VBI\ .\ VAC\ .\ \tan\alpha} , \tag{51}$$

where α is a given aiming angle.

Let n = Tpb and then the multiple hit probability (PMH) is:

$$PMH = 1 - e^{-\frac{N_{,}A}{2\pi\sigma^2}} \quad (52)$$

where A is total projected target's area.

By using this model (equation (52), we will show an example in Tables 5, 5-1, 5-2, 503, 5-4, 5-5, and 5-6, and in Figures 12-1, 12-2, 12-3, 12-4, 12-5, 12-6, and 13. We will consider multiple-shot hit probabilities (when the gun is moving with the moving target) in the next section.

Table 5 : Example of hit probabilities when 9 people are firing at one aircraft.

Model Aircraft	:	MIG-23,
Model Gun	:	50-Calibers (MGS),
Aircraft Area (A)	:	826 ft,
Ratio of Fuselage (b)	:	.27 ,
Wing Span (WS)	:	4210 ft,
Length (LN)	:	55.0 ft,
Height (HT)	:	6.19 ft,
Dispersion angle (θ)	:	1 ,
Aircraft speed (VAC)	:	733 ft/sec (500 MPH),
Mussle velocity (VBI)	:	3000 ft/sec,
Bullet round (RD)	:	10 RD/sec.

Out put (PSH) and (PMH) are as shown in Table 5-1 and 5-6. We want to fire with 9 people because 9 people are one military element unit. We will show the hit probabilities as the aiming angle is changed. This is done in Figure (12-1 and 12-6) and total number of intercepting bullets is shown in Figure 13 (Tpb)

Table 5-1 : Hit probabilities when gun is fixed at a certain aiming angle, and when 9 people are firing at one aircraft

ALTITUDE= 500.00

ALPA	AREA	RS	TPB	PSH	PMH
19.99990	475.36021	3689.10474	10.66717	0.10969	0.71710
22.99997	488.21157	3675.05394	10.46127	0.14419	0.80386
25.99995	506.01440	3659.11279	10.13283	0.18387	0.87239
28.99994	527.08043	3641.38525	9.85667	0.22867	0.92203
31.99992	550.64712	3621.89941	9.61708	0.27584	0.95512
34.99991	574.02148	3600.70923	9.40382	0.32604	0.97554
37.99989	598.38037	3577.87280	9.20990	0.37744	0.98728
40.99950	622.65308	3553.45215	9.03044	0.42878	0.99363
43.99930	645.47510	3527.51465	8.86195	0.47391	0.99690
46.99935	669.55444	3500.13110	8.70185	0.52683	0.99851
49.99933	691.05137	3471.37695	8.54824	0.57176	0.99929
52.99982	712.56089	3441.33081	8.39971	0.61312	0.99966
55.99900	732.13057	3410.07495	8.25516	0.65050	0.99983
58.99977	750.19580	3377.69531	8.11378	0.68390	0.99991
61.99976	766.03047	3344.23027	7.97495	0.71315	0.99995
64.99974	781.34100	3309.72163	7.83820	0.73842	0.99997
67.99973	794.21313	3274.71333	7.70317	0.75968	0.99998
70.99971	805.10655	3238.75220	7.56962	0.77777	0.99999
73.99969	814.12091	3202.13672	7.43735	0.79234	0.99999
76.99900	821.03407	3164.90729	7.30624	0.80382	0.99999
79.99966	825.62764	3127.34570	7.17620	0.81241	0.99999
82.99954	828.40143	3089.37549	7.04718	0.81829	0.99999
85.99940	828.85624	3051.16138	6.91917	0.82156	0.99999
88.99925	827.10010	3012.80762	6.79217	0.82230	0.99999
91.99910	827.94727	2974.41797	6.66617	0.82234	0.99999
94.99896	826.99730	2936.07839	6.54121	0.82076	0.99999
97.99881	827.62715	2897.95483	6.41730	0.81663	0.99998
100.99867	824.46757	2860.09309	6.29446	0.80986	0.99997
103.99852	818.96875	2822.60962	6.17270	0.80033	0.99995
106.99837	811.37280	2785.61450	6.05202	0.78765	0.99992
109.99823	801.74072	2749.20703	5.93237	0.77220	0.99985
112.99808	790.14062	2713.48755	5.81373	0.75315	0.99971
115.99794	776.64941	2678.55273	5.69598	0.73045	0.99943
118.99700	761.35069	2644.49878	5.57898	0.70368	0.99897
121.99773	744.36328	2611.41846	5.46253	0.67327	0.99778
124.99757	725.78504	2579.40454	5.34636	0.63856	0.99566
127.99742	705.75510	2548.54248	5.23007	0.59979	0.99168
130.99726	684.42920	2518.91840	5.11315	0.55720	0.98448
133.99713	661.96218	2490.61230	4.99490	0.51120	0.97199
136.99699	638.62542	2463.70239	4.87439	0.46245	0.95147
139.99684	614.62231	2438.25245	4.75036	0.41180	0.91962
142.99669	590.27686	2414.36182	4.62109	0.36031	0.87313
145.99655	565.98828	2392.06616	4.48424	0.30917	0.80959
148.99640	542.27222	2371.43701	4.33647	0.25964	0.72846
151.99625	519.62666	2352.53027	4.17299	0.21294	0.63134
154.99611	499.03940	2335.39644	3.98665	0.17316	0.52459
157.99597	483.16115	2320.06765	3.76637	0.13214	0.41362
160.99582	472.78760	2306.64014	3.49399	0.09945	0.30650

ALPA(α) : Aiming angle,

AREA(A) : Projected area,

RS : Relative speed,

TPB(N) : Total intercepting bullets,

Psh : Single-shot hit probabilities,

Pmh : Multiple-shot hit probabilities.

Table 5-6 : Hit probabilities when gun is fixed at a certain aiming angle, and when 9 people are firing at one aircraft.

ALTITUDE= 1000.00

ALPA	AREA	RS	TPB	PSH	PMH
19.99998	475.36621	3689.10474	10.86717	0.02863	0.27070
22.99997	483.11157	3675.03394	10.46127	0.03813	0.33451
25.99995	506.01440	3659.11279	10.13283	0.04953	0.40232
28.99994	527.08643	3641.38525	9.85667	0.06266	0.47158
31.99992	550.04712	3621.84941	9.61708	0.07752	0.53974
34.99991	574.02148	3600.70923	9.40382	0.09394	0.60452
37.99989	598.38037	3577.37280	9.20990	0.11173	0.66418
40.99986	622.05368	3553.45215	9.03044	0.13064	0.71754
43.99985	646.47510	3527.51465	8.86195	0.15037	0.76404
46.99985	669.55444	3500.13110	8.70165	0.17062	0.80366
49.99983	691.65137	3471.37055	8.54824	0.19105	0.83673
52.99982	712.56689	3441.33081	8.39971	0.21133	0.86387
55.99980	732.13037	3410.07495	8.25516	0.23115	0.88581
58.99977	750.19580	3377.69531	8.11378	0.25018	0.90330
61.99976	766.03647	3344.28027	7.97695	0.26817	0.91707
64.99974	781.34106	3309.92163	7.83870	0.28484	0.92776
67.99973	794.21313	3274.71338	7.70317	0.29958	0.93590
70.99971	805.16699	3238.75220	7.56962	0.31341	0.94194
73.99969	814.12871	3202.13672	7.43735	0.32495	0.94621
76.99968	821.03467	3164.96729	7.30624	0.33448	0.94895
79.99966	825.82764	3127.34570	7.17620	0.34139	0.95033
82.99954	826.46143	3089.37549	7.04710	0.34710	0.95043
85.99940	828.85024	3051.16135	6.91917	0.35007	0.94928
88.99925	827.10010	3012.80762	6.79217	0.35074	0.94680
91.99910	827.94727	2974.41797	6.66617	0.35077	0.94384
94.99890	828.95730	2936.09839	6.54121	0.34933	0.93936
97.99881	827.62715	2897.95483	6.41730	0.34561	0.93421
100.99857	824.40997	2860.09009	6.29446	0.33966	0.92663
103.99852	818.96875	2822.60962	6.17270	0.33154	0.91677
106.99837	811.37280	2785.61450	6.05202	0.32133	0.90423
109.99823	801.74072	2749.29703	5.93237	0.30914	0.88852
112.99808	790.14062	2713.48755	5.81373	0.29513	0.86910
115.99794	776.04941	2678.55273	5.69598	0.27945	0.84539
118.99766	761.35009	2644.49878	5.57898	0.26232	0.81684
121.99773	744.36326	2611.41846	5.46253	0.24356	0.78295
124.99757	725.78564	2579.40454	5.34636	0.22463	0.74339
127.99742	705.75610	2548.54248	5.23007	0.20463	0.69802
130.99728	684.42720	2518.91846	5.11315	0.18426	0.64702
133.99715	661.93218	2490.61230	4.99490	0.16385	0.59092
136.99699	638.02842	2463.70239	4.87439	0.14374	0.53066
139.99684	614.62231	2438.26245	4.75036	0.12425	0.46754
142.99669	590.27036	2414.36182	4.62109	0.10566	0.40318
145.99655	565.98826	2392.06616	4.48424	0.08832	0.33942
148.99640	542.27222	2371.43701	4.33647	0.07240	0.27813
151.99626	519.22000	2352.53027	4.17299	0.05811	0.22105
154.99611	499.63940	2335.39844	3.98665	0.04556	0.16964
157.99597	483.18115	2320.08765	3.76637	0.03481	0.12493
160.99582	472.78760	2306.64014	3.49399	0.02585	0.08744

ALPA(α) : Aiming angle,

AREA(A) : Projected area,

RS : Relative speed,

TPB(N) : Total interceptiong bullets,

Psh : Single-shot hit probabilities,

Pmh : Multiple-shot hit probabilities.

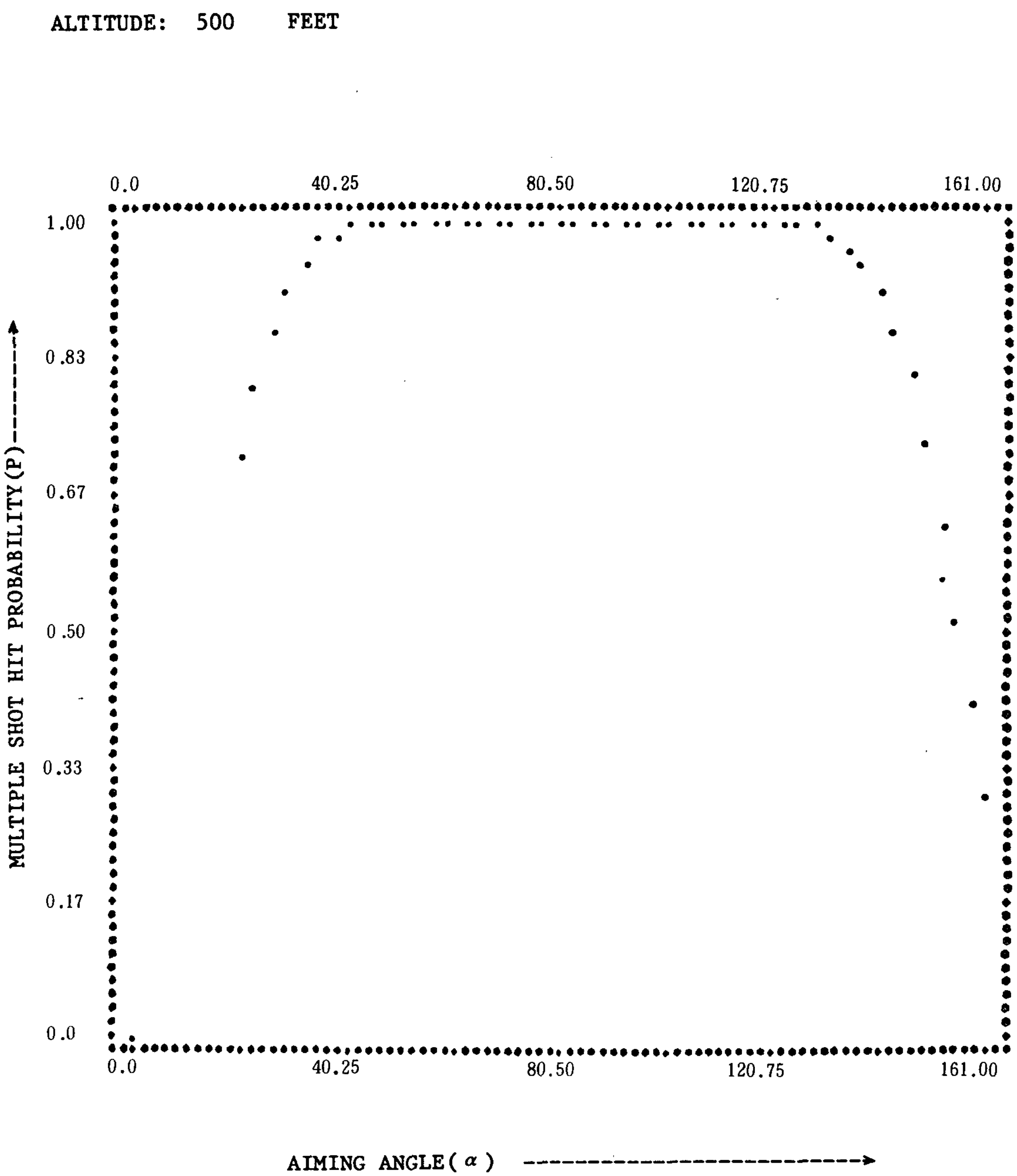

Figure 12-1 : Multiple shot hit probability (9 people) for a target at 500 feet altitude.

ALTITUDE: 1000 FEET

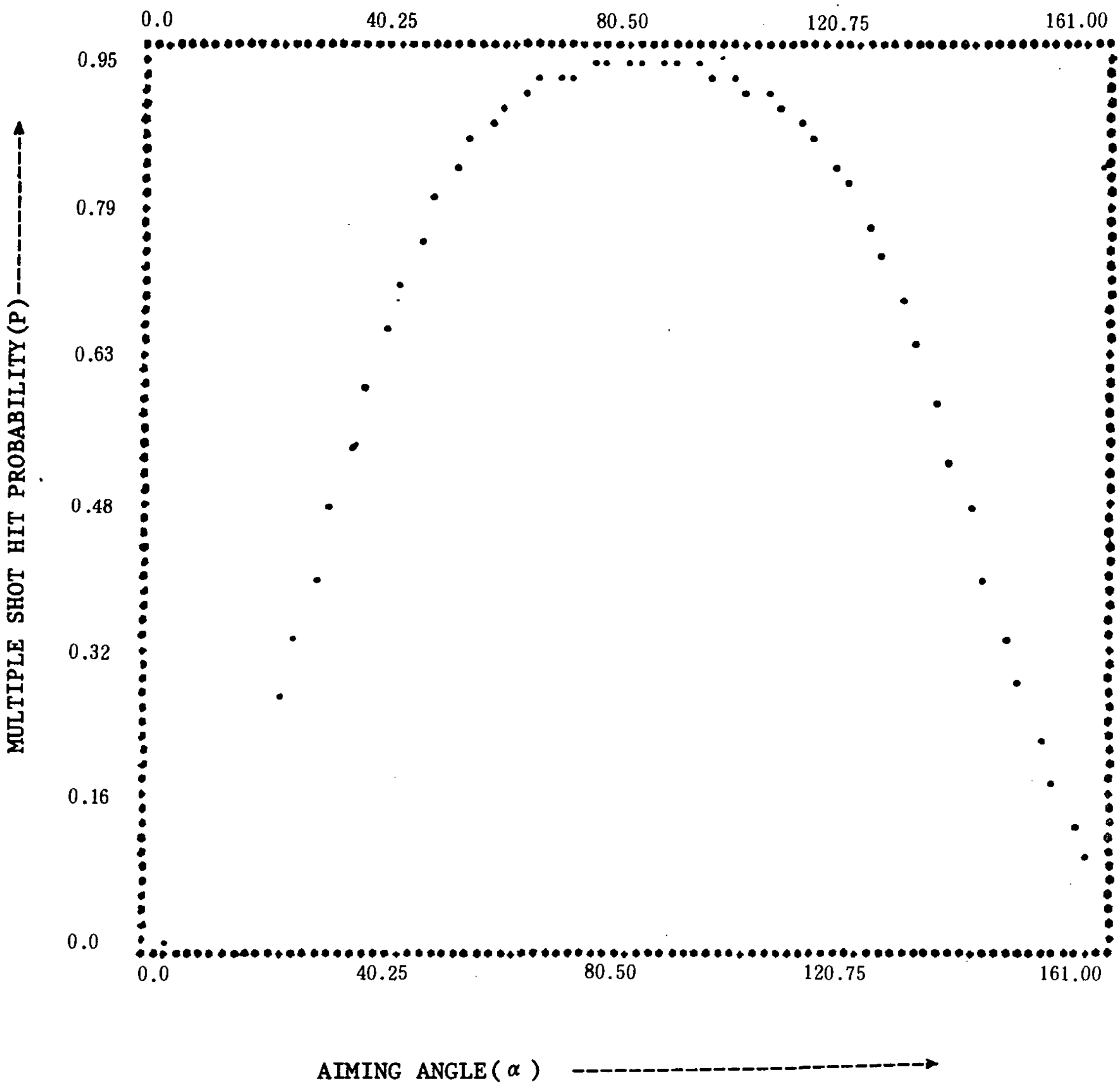

Figure 12-6 : Multiple shot hit probability (9 people) for a target at 1000 feet altitude.

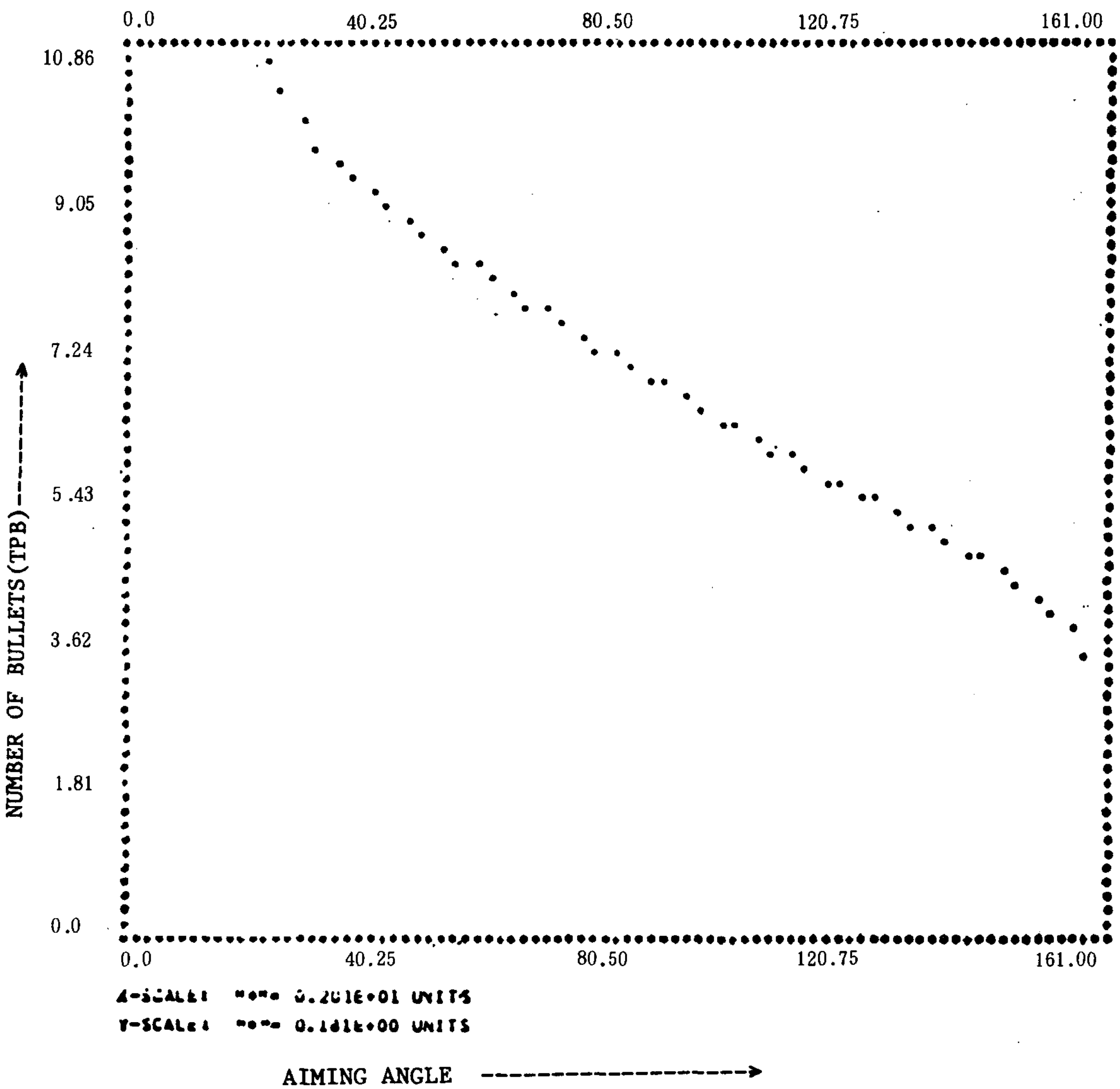

Figure 13 : Total number of intercepting bullets during one over flight .

2 . The case when the gun is moving with the aircraft .

In this section, we will discuss the multiple-shot hit probabilities when gun is moving with a certain lead angle relative to the aircraft, as shown in Figure 14 (Geometry of the Overflight). When aircraft appears suddenly, the gunner is assumed to fire using a certain lead angle and the gunner will follow the aircraft. We will consider the maximum gun range which depends on the altitutde, and the angle between gun and aircraft. We will use the angle (between gun-target line and the horizontal line) that is from 20 degrees to 160 degrees, from 30 degrees to 150 degrees and from 40 degrees to 140 degrees.

Figure 14 illustrates that gunner's fire will move with the aircraft.

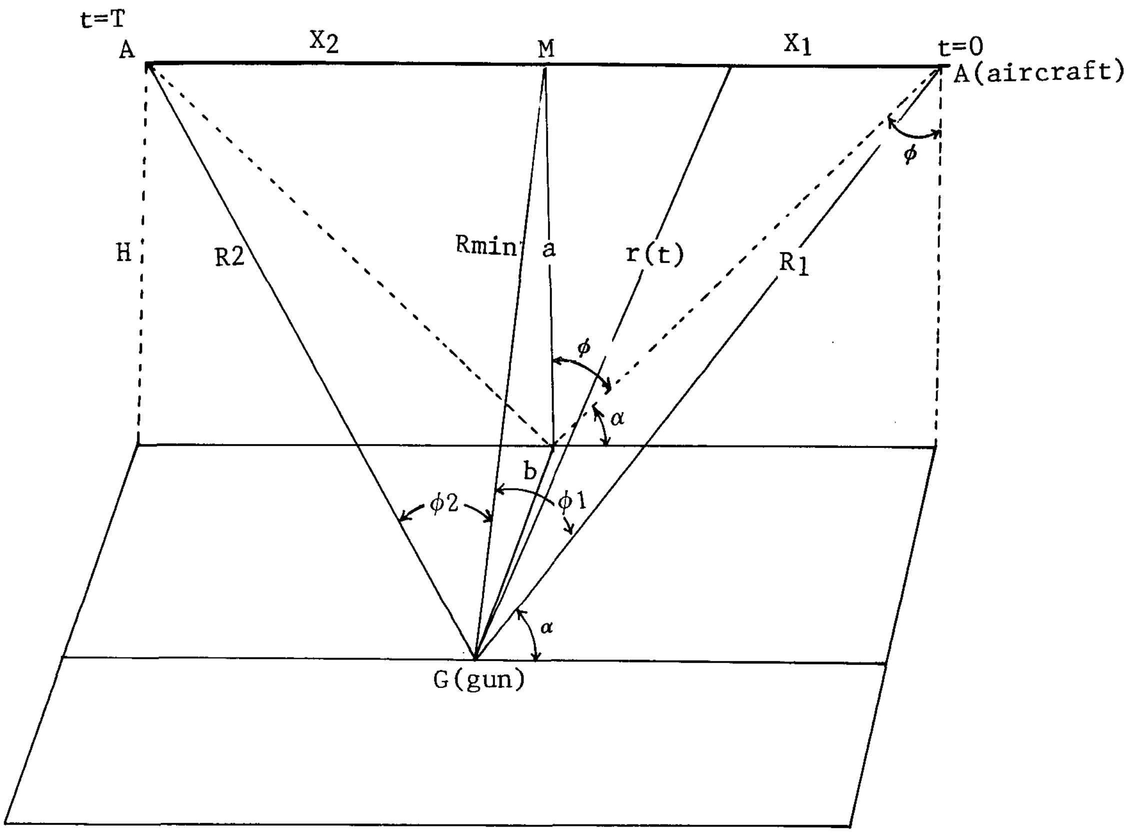

Figure 14. Geometry of the Overflight when gun is moving with the aircraft.

$$R\ min = \sqrt{a^2 + b^2}$$

if B=0, then aircraft flies over head

G : Gun position

A,A' : Aircraft position

$\phi_1 + \phi_2$: Total Gun moving angle

R_1, R_2 : Gun's maximum firing range

(1) Mean single shot hit probability (PSH) averaged over entire pass.

The mean value of PSH, with the effect of slant range averaged out, is found from:

$$\bar{P} = \frac{1}{T} \int_0^T Psh \cdot r(t)\, dt \quad . \tag{53}$$

From Figure 14:

$$r^2(t) = x(t)^2 + R^2 min \quad , \tag{54}$$

$$x(t) = X_1 - Vac \cdot t \quad ,$$

and from equation (44)

$$PSH = 1 - e^{-\frac{A}{2\pi\sigma^2}}$$

Let $A = \pi R^2$ · then

$$PSH = 1 - e^{-\frac{R^2}{2\pi\sigma^2}} \quad ,$$

and from the Taylor series

$$e^{-\frac{R^2}{2\sigma^2}} = 1 - \frac{R^2}{2\sigma^2} \quad , \text{ when R is small.}$$

$$\text{therefore, } PSH = \frac{R^2}{2\sigma^2} \tag{55}$$

From equation (53), (54) and (55), we have

$$\bar{P} = \frac{1}{T} \cdot R^2/2 \cdot \quad ((x_1 - Vac \cdot t)^2 + R^2 min)\, dt$$

$$= \frac{R^2 \; (\phi 1 + \phi 2)}{2\sigma^2 \; Rmin \; (x_1 + x_2)}$$

Let $R^2 \doteq \frac{A}{\pi}$, $\phi = \phi 1 + \phi 2$ and $X = X_1 + X_2$.

Then

$$\bar{P} = A \cdot (\phi) / \; 2\pi\sigma^2 \; Rmin \cdot X$$

where $\phi 1 = \phi 2$, $x_1 = x_2$, and R min = Altitude (H) because we assume that aircraft moves directly over head.

Thus

$$\boxed{P = A \cdot (\phi) / 2\pi\sigma^2 \quad H \cdot x} \tag{56}$$

from Figure 14;

$$\phi 1 = \tan^{-1}(x_1/Rmin) = (\cos^{-1}(Rmin/R_1)$$

$$\phi 2 = \cos^{-1}(Rmin/R_2) \quad , \text{where } \phi_1 = \phi_2 , \tag{57}$$

$R_1 = H/\sin\alpha$ (angle α is dependent on maximum range),

and

R min = H where b = 0 .

From equation (57) and Figure 14,

$$\phi = 2 \cos^{-1}(H/\sin\alpha) ,$$

$$\cos(.5\,\phi) = \sin\alpha ,$$

$$\boxed{\phi = \pi - 2\alpha} , \tag{58}$$

$$x_1 = H / \tan\alpha ,$$

$$x = 2x_1 = 2H / \tan\alpha \tag{59}$$

and therefore,

$$\bar{P} = \frac{A \cdot (\pi - 2\alpha)}{2\pi\sigma^2 \cdot H \cdot 2H/\tan\alpha} ,$$

or

$$\boxed{\bar{P} = \frac{A \cdot (\pi - 2\alpha) \cdot \tan\alpha}{4\pi \cdot H^2\sigma^2}} \tag{60}$$

where = 3.14159, α is in radian.

Therefore, from equation (62) and (63), we have

$$Pmh = 1 - EXP\left(-\left(\frac{N \cdot 2H}{Vac \cdot \tan\alpha}\right)\left(\frac{A(\pi - 2\alpha)\tan\alpha}{4\pi H^2\sigma^2}\right)\right),$$

or

$$Pmh = 1 - EXP\left(-\frac{N \cdot A(\pi - 2\alpha)}{2\pi\sigma^2 \cdot 4H \cdot VAC}\right), \tag{60-1}$$

where N is intercepting bullets,

A is projected area,

α is aiming angle,

H is alititude,

VAC is aircraft's speed,

σ is standard deviation,

(as $\sigma = d_o - d_1$ VAC)

and d_o, d_1 are as shown in Table 6.

Table 6 : The Factors for Standard Deviation

weapon Type / Factor	I	II	III	IV	V
d_o	.00418	.01188	0	0	.01180
d_1	.000357	.000276	.000287	.000296	.0000608

where I : M-1 Rifles

II : Browning Automatic Rifles (BAR)

III : Single 50-Caliber Machine Gun (MG)

IV : Quad - 50 - Caliber Machine Guns

V : Twin 40mm Guns

These factors are used following formulation as

$$\sigma = d_0 + d_1 VAC ,$$

where VAC is aircraft's speed.

(2) Multiple-hit probability for the entire pass.

Let the total passing time (Tp) be T, where

$$T = \frac{X}{VAC}$$

from equation (59),

$$T = \frac{2H}{VAC \cdot \tan\alpha} \text{ (sec)} . \qquad (61)$$

Let the gun's rate of firing be

$$q = N \text{ Rd/Sec}$$

Then, total intercepting bullets during time T is

$$M = q.T = \frac{N \cdot 2H}{Vac \cdot Tan\,\alpha} \qquad (62)$$

Using the average probability (60), multiple-shot hit probability is

$$PMH = 1 - e^{-M\bar{P}} \qquad (63)$$

If S people are firing during time interval T, then total intercepting bullets (Tn) is

$$TN = S.N . \qquad (64)$$

Then we have from equation (60-1) and equation (64),

$$PMH = EXP\left(- \frac{S.N.A\ (\pi - 2\alpha)}{2\pi\sigma^2 \cdot H \cdot VAL} \right) \qquad (65)$$

Using this multiple-shot hit probability model for the oase when the gun is moving with the aircraft, we will display an example in

Table 7 (7-1, 7-2). The FORTRAN program is given in Appendix B.

Now we have done single-shot hit probability, and multiple-shot hit probabilities. In Appendix C, we will study kill probability using the lethal area that is different for each weapon.

Table 7 : Example of multiple-shot hit probabilities when gun is moving with the aircraft.

Model Aircraft	:	MIG-23
Initial aiming angle (α)	:	20 , 30 , 40
Weapon Type IV	:	quad to Calibers
Reciprocal Gun Rate (N)	:	10 Rd/Sec
Altitude (H)	:	500 ft - 1000 ft
Aircraft Speed (VAC)	:	733 ft/sec(500MPH)
Factor of Standard Deviation (d)	:	0.000296
Area (A)	:	626.0 ft

Output is as shown in Tables 7-1, 7-2,

Table 7-1; hit probabilities when gun is moving with the aircraft as aiming angle changes 20, S people are 1, 3, 5, 7, 9, intercepting bullets (N) is 10RD/sec, aircraftspeed is 733 ft/sec, bullet's speed is 3250 ft/sec, projected area (A) is 626 ft^2.

ALTITUDE(H)	P1	P3	P5	P7	P9
500.00000	0.26159	0.59738	0.78047	0.88030	0.93473
600.00000	0.22331	0.53146	0.71735	0.83949	0.89714
700.00000	0.19475	0.47786	0.66144	0.78047	0.85765
800.00000	0.17265	0.43368	0.61235	0.73466	0.81837
900.00000	0.15505	0.39675	0.56931	0.69251	0.78047
1000.00000	0.14069	0.36547	0.53146	0.65402	0.74453

Pi: multiple-shot hit probabilities

where i = 1, 3, 5, 7, 9 (people)

Table 7-2; hit probabilities when gun is moving with the aircraft. Initial aiming angle (α) is 30°.

ALTITUDE	P1	P3	P5	P7	P9
500.00000	0.22889	0.54150	0.72737	0.83789	0.90361
600.00000	0.19475	0.47786	0.66144	0.78047	0.85765
700.00000	0.16945	0.42707	0.60478	0.72737	0.81194
800.00000	0.14995	0.38576	0.55616	0.67928	0.76825
900.00000	0.13446	0.35158	0.51423	0.63609	0.72737
1000.00000	0.12187	0.32287	0.47786	0.59738	0.68954

Pi: multiple-shot hit probabilities,

where i = 1, 3, 5, 7, 9 (people)

Ⅲ. CONCLUSION AND RECOMMENDATION

The models developed in Section II and III are based on certain assumptions. These create idealized conditions but should still be indicative of real situations and capable of yielding useful results.

As shown in the multiple-hit probability model when the gun is fixed at aiming angle these probabilities depend on the relative speed between aircraft speed and the gun speed, and single-shot hit probability depends on the aircraft projected area and altitude. We know that single-shot hit probabilities increase as angle (α) increases up to 90 degress. After 90 degrees, these probabilities will decrease as shown in Table 5. Also, multiple-shot hit probabilities increase up to 83 degrees, and decrease after 84 degrees.

The best fixed angle is 83 degrees, as shown in Table 5. As shown Table 5, we can see that hit probabilities decrease as altitude increases, and standard error increases.

The multiple hit probabilities when the gun is fixed at aiming angle (83 degree) are greater than when the gun is moving with the aircraft. Here, the hit probability is low when one person is firing at the moving target.

An extension of this model by taking into account the wind velocity and air density should produce better results because the bullets's and aircraft's speed are affected by wind and air density. Another useful possible extension of this model would be the generation of hit probabilities if the aircraft are moving tangentially to a gunner's position. Also, this model could be extented to the generation of multiple-shot hit probabilities when several people fire at several aircraft.

Finally these models should be useful for Army, Airforce, Navy and Marine applications.

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