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ON THE H-FUNCTION OF n-VARIABLES II

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1. Introduction

Following the author [5, p.221] the H-function of several complex variables is defined in terms of a multiple integral of Mellin-Barnes type as follows:

$$(1.1)\quad H^{A,(M_n:N_n)}_{C,D,(P_n:Q_n)}\left[\begin{array}{c|l} x_1 & (c_j^*,\ \gamma_j^{(r)}),\ (d_j,\ \delta_j^{(r)}) \\ \vdots & \\ x_n & (a_j^{(r)},\ \alpha_j^{(r)}),\ (b_j^{(r)},\ \beta_j^{(r)}) \end{array}\right]$$

$$=H^{A,(M_n:N_n)}_{C,D,(P_n:Q_n)}\left[\begin{array}{c|l} x^1 & (c_1^*,\ \gamma_1^{(1)},\ \cdots,\ \gamma_1^{(n)}),\ \cdots,\ (c_C^*,\ \gamma_C^{(1)},\ \cdots,\ \gamma_C^{(n)}) \\ \vdots & (d_1,\ \delta_1^{(1)},\ \cdots,\ \delta_1^{(n)}),\ \cdots,\ (d_D,\ \delta_D^{(1)},\ \cdots,\ \delta_D^{(n)}) \\ \vdots & (a_{P_1}^{(1)},\ \alpha_{P_1}^{(1)}),\ \cdots,\ (a_{P_n}^{(n)},\ \alpha_{P_n}^{(n)}) \\ x^n & (b_{Q_1}^{(1)},\ \beta_{Q_1}^{(1)}),\ \cdots,\ (b_{Q_n}^{(n)},\ \beta_{Q_n}^{(n)}) \end{array}\right]$$

$$=\frac{1}{(2\pi i)^n}\int_{-i\infty}^{i\infty}\cdots(n)\int_{-i\infty}^{i\infty}\phi_1^*(S_1,\ \cdots,\ S_n)\prod_{r=1}^{n}\{\phi_2^*(S_r)(x_r)^{S_r}(dS_r)\}$$

where $i=(-1)^{\frac{1}{2}}$,

$$(1.2)\quad \phi_1^*(S_1,\ \cdots,\ S_n)=\frac{\prod_{j=1}^{A}\Gamma\left(c_j^*+\sum_{r=1}^{n}S_r\gamma_j^{(r)}\right)}{\prod_{j=A+1}^{C}\Gamma\left(1-c_j^*-\sum_{r=1}^{n}S_r\gamma_j^{(r)}\right)\prod_{j=1}^{D}\Gamma\left(d_j+\sum_{r=1}^{n}S_r\delta_j^{(r)}\right)},$$

$$(1.3)\quad \phi_2^*(S_r)=\left\{\frac{\prod_{j=1}^{M_r}\Gamma(b_j^{(r)}-S_r\beta_j^{(r)})\prod_{j=1}^{N_r}\Gamma(1-a_j^{(r)}+S_r\alpha_j^{(r)})}{\prod_{j=M_r+1}^{Q_r}\Gamma(1-b_j^{(r)}+S_r\beta_j^{(r)})\prod_{j=N_r+1}^{P_r}\Gamma(a_j^{(r)}-S_r\alpha_j^{(r)})}\right\}$$

where $z_j\neq 0$ $(j=1,\ \cdots,\ n)$ and an empty product is interpreted as unity.

Further $A,\ C,\ D,\ M_1,\ \cdots,\ M_n:N_1,\ \cdots,\ N_n:P_1,\ \cdots,\ P_n$ and $Q_1,\ \cdots,\ Q_n$ are non-negative integers, satisfying the inequalities $0\leq A\leq C$, $1\leq M_j\leq Q_j$, $0\leq N_j\leq P_j$ $(j=1,\ \cdots,\ n)$; $a_j^{(r)}$, s, $b_j^{(r)}$, s, $c_j^{*'}$ s and d_j's are all complex numbers and α's, β's, γ's, and δ's are all positive numbers. The poles of the integrand of (1.1) are simple. The paths of integration are indented, if necessary, to ensure that all the poles of $\Gamma\left(b_j^{(r)}-\sum_{r=1}^{n}S_r\beta_j^{(r)}\right)$ for $j=1, \cdots, M_r$ $(r=1, \cdots, n)$ are separated from the poles of $\Gamma\left(c_j^*+\sum_{r=1}^{n}S_r\gamma_j^{(r)}\right)$ for $j=1, \cdots, A$ and $\Gamma\left(1-a_j^{(r)}+\sum_{r=1}^{n}S_r\alpha_j^{(r)}\right)$ for $j=1, \cdots, N_r$ $(r=1, \cdots, n)$.

The function represented by the integral (1.1) is an analytic function of $x_1, \cdots, x_n$ provided that

$$(1.4)\quad \lambda_r=\sum_{j=1}^{C}\gamma_j^{(r)}+\sum_{j=1}^{P_r}\alpha_j^{(r)}-\sum_{j=1}^{D}\delta_j^{(r)}-\sum_{j=1}^{Q_r}\beta_j^{(r)}<0 \text{ for } r=1, \cdots, n.$$

It can be readily seen from the asymptotic expansion of the gamma function that the integral (1.1) converges absolutely provided that

$$(1.5)\qquad |\arg x_r|<\frac{\pi\mu_r}{2}\quad (r=1, \cdots, n)$$

where

$$(1.6)\quad \mu_r=\sum_{j=1}^{N_r}\alpha_j^{(r)}+\sum_{j=1}^{M_r}\beta_j^{(r)}-\sum_{j=A+1}^{C}\gamma_j^{(r)}-\sum_{j=1}^{D}\delta_j^{(r)}-\sum_{j=M_r+1}^{Q_r}\beta_j^{(r)}-\sum_{j=N_r+1}^{P_r}\alpha_j^{(r)}>0$$

In earlier papers [4 and 5] the author has obtained certain properties of the H-function of several complex variables. In the present paper we evaluate two integrals associated with this function. The first integral gives the Euler transform of this function. The second integral enables us to compute the Weyl (Fractional) integral of order ρ of the product of an exponential function and the H-function of several complex variables. Finally an expansion formula is developed for this function in a series of a specialized Fox's H-function and a related H-function of n-variables. The results obtained are believed to be new.

2. Integrals

The first integral to be proved here is

$$(2.1)\quad \int_0^1 t^{\tau-1}(1-t)^{\omega-1}\, H_{C,D,(P_n:Q_n)}^{0,(M_n:N_n)}\begin{bmatrix} x_1t^{\rho_1}(1-t)^{\sigma_1} \\ \vdots \\ x_nt^{\rho_n}(1-t)^{\sigma_n}\end{bmatrix} dt$$

$$=H_{C+2,D,(P_n:Q_n)}^{2,(M_n:N_n)}\left[\begin{matrix} x_1 \\ \vdots \\ x_n\end{matrix}\;\middle|\;\begin{matrix} (\tau, \rho_1, \cdots, \rho_n), (\omega, \sigma_1, \cdots, \sigma_n), (c_j^*, \gamma_j^{(r)}) \\ ; (\omega+\tau, \rho_1, \cdots, \rho_n, \sigma_1, \cdots, \sigma_n), (d_j, \delta_j^{(r)}); \\ (a_j^{(r)}, \alpha_j^{(r)}); (b_j^{(r)}, \beta_j^{(r)})\end{matrix}\right]$$

where $R[\tau+\rho_1 \min(b_{j_1}/\beta_{j_1})+\cdots+\rho_n \min(b_{j_n}/\beta_{j_n})]>0$,

$R[\omega+\sigma_1 \min(b_{j_1}/\beta_{j_1})+\cdots+\sigma_n \min(b_{j_n}/\beta_{j_n})]>0$ for $j_1=1, \cdots, M_1, \cdots, j_n=1, \cdots, M_n$, $|\arg x_j| <\frac{1}{2}\pi\mu_j$, $\mu_j>0$ for $j=1, \cdots, n$.

(2.1) can be readily established on using (1.1) and the beta function formula. In a similar manner if we employ the integral

$$(2.2)\qquad \int_n^\infty e^{-\theta t}(t-x)^{\rho-1}\,dt=\theta^{-\rho}e^{-\theta x}\Gamma(\rho) \text{ where } R(\theta x)>0,\ R(\rho)>0,$$

we find that

$$(2.3)\qquad \int_x^\infty e^{-\theta t}(t-x)^{\rho-1} H^{A,(M_n:N_n)}_{C,D,(P_n:Q_n)}\begin{bmatrix} x_1(t-x)^{\sigma_1} \\ x_n(t-x)^{\sigma_n}\end{bmatrix} dt$$

$$=e^{-\theta x}H^{A+1,(M_n:N_n)}_{C+1,D,(P_n:Q_n)}\left[\begin{matrix} x_1/\theta^{\sigma_1} \\ \vdots \\ x_n/\theta^{\sigma_n}\end{matrix}\left|\begin{matrix}(\rho,\sigma_1,\cdots,\sigma_n),(c_j^*,\gamma_j^{(r)}), \\ (d_j,\delta_j^{(r)}) \\ (a_j^{(r)},\alpha_j^{(r)}),(b_j^{(r)},\beta_j^{(r)})\end{matrix}\right.\right]$$

where $R(\theta x)>0$, $R[\rho+\sigma_1 \min(b_{j_1}/\beta_{j_1})+\cdots+\sigma_n \min(b_{j_n}/\beta_{j_n})]>0$ for $j_1=1, \cdots, M_1, \cdots, j_n=1, \cdots, M_n$, $|\arg x_j|<(\pi/2)\mu_j$, $\mu_j>0$.

On using the identity [5. p. 223, eq. (2.3)], the following results can be easily deduced from (2.1) and (2.3).

$$(2.4)\qquad \int_0^1 t^{\tau-1}(1-t)^{\omega-1}\prod_{j=1}^n H^{M_j,N_j}_{P_j,Q_j}\left[x_j t^{\rho_j}(1-t)^{\sigma_j}\left|\begin{matrix}(a_{P_j}^{(j)},\alpha_{P_j}^{(j)}) \\ (b_{Q_j}^{(j)},\beta_{Q_j}^{(j)})\end{matrix}\right.\right]dt$$

$$=H^{2,(M_n:N_n)}_{2,0,(P_n:Q_n)}\left[\begin{matrix} x_1 \\ \vdots \\ x_n\end{matrix}\left|\begin{matrix}(\tau,\rho_1,\cdots,\rho_n),(\omega,\sigma_1,\cdots,\sigma_n) \\ (\omega+\tau,\rho_1,\cdots,\rho_n,\sigma_1,\cdots,\sigma_n) \\ (a_j^{(r)},\alpha_j^{(r)}),(b_j^{(r)},\beta_j^{(r)})\end{matrix}\right.\right].$$

Under the same conditions of validity as given with (2.1).

$$(2.5)\qquad \int_x^\infty e^{-\theta t}(t-x)^{\rho-1}\prod_{j=1}^n\left\{H^{M_j,N_j}_{P_j,Q_j}\left[x_j(t-x)^{\sigma_j}\right]\right\}dt$$

$$=e^{-\theta x}H^{1,(M_n:N_n)}_{1,0,(P_n:Q_n)}\left[\begin{matrix} x_1/\theta^{\sigma_1} \\ \vdots \\ x_n/\theta^{\sigma_n}\end{matrix}\left|\begin{matrix}(\rho:\sigma_1,\cdots,\sigma_n); \\ (a_j^{(r)},\alpha_j^{(r)}),(b_j^{(r)},\beta_j^{(r)})\end{matrix}\right.\right]$$

under the same sets of conditions of validity as given with (2.3).

3. Expansion formula.

The expansion formula to be proved here is

$$(3.1)\qquad x^{\sigma} H^{A,\ (M_n:N_n)}_{C,D,\ (P_n:Q_n)}\begin{bmatrix} x\rho_1 \\ \vdots \\ x\rho_n \end{bmatrix}$$

$$=\sum_{K=0}^{\infty}(2\lambda+2K)\ H^{\rho,\tau+1}_{\nu+2,\tau}\left[\frac{1}{x}\ \middle|\ \begin{matrix}(1-\lambda-K,1),\ (e_\nu, E_\nu),\ (1+\lambda+K,1)\\ (f_\rho,\ F_\rho)\end{matrix}\right]$$

$$\times H^{A+\nu-\tau+1,\ (M_n:N_n)}_{c+\nu+2,\ D+\rho,\ (P_n:Q_n)}\left[\begin{matrix}\rho_1\\ \vdots\\ \rho_n\end{matrix}\ \middle|\ \mathscr{R}^*\right]$$

where $|\arg x\rho_j|<\frac{1}{2}\pi\mu_j,\ \mu_j>0,\ |\arg x|<\frac{1}{2}\pi\Omega$;

$$\Omega=\sum_{j=1}^{\rho}F_j-\sum_{j=1}^{\tau}E_j-\sum_{j=\tau+1}^{\nu}E_j>0,$$

$$\mathscr{R}^*=\begin{bmatrix}(\sigma+\lambda+K,1),\ (c_j^*,\ \gamma_j^{(r)}),\ (e_{\tau+1}+E_{\tau+1}\sigma,\ E_{\tau+1}),\ \cdots,\\ (e_\nu+E_\nu\sigma,\ E_\nu),\ (e_1+E_1\sigma,\ E_1)\ \cdots,\ (e_\tau+E_\tau\sigma,\ E_\tau),\\ (\sigma-\lambda-K,\ 1),\ (d_j,\ \delta_j^{(r)}),\ (f_\rho+F_\rho\sigma,\ F_\rho)\\ (a_j^{(r)},\ \alpha_j^{(r)});\ (b_j^{(r)},\ \beta_j^{(r)})\end{bmatrix}$$

and the series on the R.H.S. converges.

(3.1) can be established with the help of (1.1) and the following expansion formula associated with a specialized Fox's H-function given recently by Benerji and Saxena [2, p.340]

$$(3.2)\qquad x^{\sigma}=\frac{\prod_{j=\tau+1}^{\nu}\Gamma(e_j+E_j\sigma)}{\prod_{j=1}^{\rho}\Gamma(f_j+\sigma F_j)\prod_{j=1}^{\tau}\Gamma(1-e_j-E_j\sigma)}\times\sum_{K=0}^{\infty}\frac{(2\lambda+2K)\Gamma(\lambda+\sigma+K)}{\Gamma(\lambda-\sigma+K+1)}$$

$$\times H^{\rho,\tau+1}_{\nu+2,\rho}\left[\frac{1}{x}\ \middle|\ \begin{matrix}(1-\lambda-K,\ 1),\ (e_\nu,\ E_\nu),\ (1+\lambda+K,\ 1),\\ (f_\rho,\ F_\rho)\end{matrix}\right]$$

As a special case of (3.1), if we set $A=C=D=0$, it is seen that

$$x^{\sigma}\prod_{j=1}^{n}\left\{H^{M_j,\ N_j}_{P_j,\ Q_j}\left[x\rho_j\ \middle|\ \begin{matrix}(a^{(j)}_{P_j},\ \alpha^{(j)}_{P_j})\\ (b^{(j)}_{Q_j},\ \beta^{(j)}_{Q_j})\end{matrix}\right]\right.$$

$$=\sum_{K=0}^{\infty}(2\lambda+2K)\ H^{\rho,\ \tau+1}_{\nu+2,\ \rho}\left[\frac{1}{x}\ \middle|\ \begin{matrix}(1-\lambda-K,1),\ (e_\nu,\ E_\nu),\ (1+\lambda+K,\ 1),\\ (f_\rho,\ F_\rho)\end{matrix}\right]$$

$$\times H^{\nu-\tau+1,\ (M_n:N_n)}_{\nu+2,\ \rho,\ (P_n:Q_n)}\left[\begin{array}{c|c}\rho_1 & \\ \vdots & \mathscr{R}^{**} \\ \rho_n & \end{array}\right]$$

where

$$\mathscr{R}^{**}=\left[\begin{array}{l}(\sigma+\lambda+K,\ 1),\ (e_{\tau+1}+\sigma E_{\tau+1},\ E_{\tau+1}),\ \cdots,\ (e_\nu+\sigma E_\nu,\ E_\nu),\\ (e_1+\sigma E_1,\ E_1),\ \cdots,\ (e_\tau+\sigma E_\tau,\ E_\tau),\ (\sigma-\lambda-K,\ 1),\\ (f_\rho+\sigma F_\rho,\ F_\rho);\ (a_j^{(r)},\ \alpha_j^{(r)});\ (b_j^{(r)},\ \beta_j^{(r)})\end{array}\right]$$

On the other hand if we take $n=2$ in (3.1), then it gives the corrected form of an expansion formula for the H-function of two variables given by Banerji and Saxena [2]

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APPENDNIX

Corrections to a paper entitled '*On a generalized function of n-variables*', Kyungpook Math. J. Vol. 14, No. 2

1. Page 255, line 4 up: Instead of $H^{(A,\ M_n:N_n)}_{B,C,\ (P_n:Q_n)}\left[\begin{array}{c}x_1\\ \vdots\\ x_n\end{array}\right]$

read $H^{A,\ (M_n:N_n)}_{C,D,\ (P_n:Q_n)}\left[\begin{array}{c}x_1\\ \vdots\\ x_n\end{array}\right]$

2. Page 256, line: Instead of

$$\left\{\prod_{j=1}^{A}\Gamma\left((c_j^*-\sum_{r=1}^{n}S_r\gamma_j\right)\right\}\Big/\left\{\prod_{j=A+1}^{C}\Gamma\left((1-c_j^*+\sum_{r=1}^{n}S_r\gamma_j\right)\right\}\left\{\prod_{j=1}^{D}\Gamma\left(d_j+\sum_{r=1}^{n}S_r\delta_j\right)\right\}$$

read

$$\left\{\prod_{j=1}^{A}\Gamma\left(c_j^*+\sum_{r=1}^{n}S_r\gamma_j\right)\right\}\Big/\left\{\prod_{j=A+1}^{C}\Gamma\left(1-c_j-\sum_{r=1}^{n}S_r\gamma_j\right)\right\}\left\{\prod_{j=1}^{D}\Gamma\left(d_j+\sum_{r=1}^{n}S_r\delta_j\right)\right\}$$

3. Page 256, line 9 up: Instead of $\Gamma\left(c_j^*-\sum_{r=1}^{n}S_r\gamma_j\right)$ read $\Gamma\left(c_j^*+\sum_{r=1}^{n}S_r\gamma_j\right)$
4. Page 256, lines 8 and 9 up: Delete "$\Gamma(b_j^{(r)}-S_r\beta_j^{(r)}$ for $j=1, \cdots, M_r$ and $r=1, \cdots, n$ are separated from the poles of"
5. Page 256, line 7 up: Insert "are separated from the poles of $\Gamma(b_j^{(r)}-S_r\beta_j^{(r)})$ for $j=1, \cdots, M_r$ and $r=1, \cdots, n$ in line 7 up after $r=1, \cdots, n$."
6. Page 258, line 10: Insert "y" before the integral.
7. Page 258, line 11: Instead of $H_{D,C}^{A,0}$ read $H_{C,D}^{A,0}$ Instead of γ_D read γ_C.
8. Page 258

   line 6 up: Instead of $R\left[\sum_{r=1}^{\infty}\{\min(b_1^{(r)}+\cdots+\min(b_j^{(r)})\}+\min(c_K/\gamma_K)+1\right]>0$

   $j=1, \cdots, S, K=1, \cdots, M$

   read $R[\min(b_{j_1}^{(1)}/\beta_{j_1}^{(1)})+\cdots+\min(b_{js}^{(S)}/\beta_{js}^{(S)})+\min(c_K^*/\gamma_K)+1]>0$

   where $j_r=1, \cdots, M_r$ $\forall r\in\{1, \cdots, S\}$ and $K=1, \cdots, A$

9. Page 258

   line 4 up: Instead of

$$\sigma=\sum_{1}^{M}\gamma_j-\sum_{j=m+1}^{Q}\gamma_j-\sum_{j=1}^{N}\gamma_j>0$$

   read

$$\sigma=\sum_{j=1}^{A}\gamma_j-\sum_{j=m+1}^{C}\gamma_j-\sum_{j=1}^{D}\delta_j>0.$$