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RIVLIN-ERICKSEN FLUID FLOW PAST A STRETCHING PLATE WITH SUCTION

By B. Siddappa and Prakash S. Hiremath

1. Introduction

In industries producing plastic films and sheets of synthetic material, one often encounters the problem of flow past a stretching sheet. Crane [1] has studied the viscos flow past a stretching plate and the nature of the boundary layer produced by the stretching plate in the ambient fluid medium. The flow in this case finds certain similarities with the Hiemenz boundary layer flow [2] near a stagnation point in which the mainstream velocity is proportional to the distance from the stagnation point. Siddappa and Khapate [3] have extended Crane's work to the Rivlin-Ericksen fluid flow and have determined the effects of non-Newtonian character of the ambient fluid medium on the boundary layer flow characteristics. In this paper, the problem of Rivlin-Ericksen fluid flow past a stretching plate with suction is studied. The velocity of the plate is assumed to increase linearly with the distance from the slit and the suction velocity at the plate is uniform and constant.

2. Formulation of the problem

A steady two-dimensional flow of Rivlin-Ericksen fluid past a porous stretching plate issuing from a slit is considered. (x, y) are rectangular cartesian coordinates with origin at the slit, x being measured along the plate in the direction of motion. (u, v) are the corresponding velocity components. Then the relevant boundary layer equations [3] are

$$uu_x+vu_y=\nu u_{yy}+\beta[(uu_x+vu_y)_{yy}+2(u_xu_y)_y]+\gamma(u_y^2)_x \tag{2.1}$$

$$\text{and } u_x+v_y=0 \tag{2.2}$$

where ν, β, γ are kinematic viscosity, visco-elasticity and cross-viscosity. The subscripts denote the partial derivatives with respect to the indicated variables.

The relevant boundary conditions are

$$\left.\begin{array}{lll} y=0\ ; & u=ax, & v=-v_0 \\ y=\infty; & u=0, & v=-c \\ & u_y=0. & \end{array}\right\} \tag{2.3}$$

where a, c, v_0 are positive constants. v_0 is the constant uniform suction velocity at the plate and c is the constant velocity with which the fluid is approaching the plate at large distance from it.

3. Solution of the problem

Set $u=axf'(y)$. (3.1)

Then eqs. (2.2) and (2.3) give $v=a[f(0)-f(y)]-v_0$ (3.2)

Now eq. (2.1) reduces to

$$f'^2+\left[f(0)-f(y)-\frac{v_0}{a}\right]f''=\frac{\nu}{a}f'''+\beta\left\{\left[f(0)-f(y)-\frac{v_0}{a}\right]f^{iv}+2f'f'''+3f''^2\right\}+2\gamma f''^2 \tag{3.3}$$

where primes denote differentiation with respect to y. The boundary conditions (2.3) reduce to

$$\begin{aligned} &y=0\ ;\ f'(0)=1\\ &y=\infty;\ f'(\infty)=0,\ \ c=v_0+a[f(\infty)-f(0)]\\ &\qquad f''(\infty)=0.\end{aligned}$$

Try a solution of the form

$$f'(y)=e^{-ky},\ \ \mathrm{Re}(k)>0. \tag{3.5}$$

Then eq. (3.3) reduces to

$$\frac{v_0\beta}{a}k^3+\left[\frac{\nu}{a}+\beta+(4\beta+2\gamma)e^{-ky}\right]k^2-\frac{v_0}{a}k-1=0. \tag{3.6}$$

Case 1. y is small.

When y is small, eq. (3.6) reduces to

$$k^3+\frac{a}{v_0\beta}\left(\frac{\nu}{a}+5\beta+2\gamma\right)k^2-\frac{1}{\beta}k-\frac{a}{v_0\beta}=0. \tag{3.7}$$

By the theory of equations [4], the roots of eq. (3.7) are real and distinct or include a complex pair according as Δ is negative or positive, where

$$108\ \Delta=-\left[\frac{4}{\beta^3}+V^2\left(\frac{A^2}{\beta^2}+18\frac{A}{\beta}-27\right)+4V^4A^3\right], \tag{3.8}$$

$$V^2=\left(\frac{a}{v_0\beta}\right)^2$$

and $A=\frac{\nu}{a}+5\beta+2\gamma$.

Now $\Delta \gtrless 0$ according as $V^2 \gtrless V_1^2$ where V_1^2 is the positive root of the quadratic equation obtained by setting eq. (3.8) equal to zero, provided that A is negative; otherwise, that is if A is positive, $\Delta \lessgtr 0$ according as $V^2 \lessgtr V_1^2$.

Thus, for small y, the components of velocity will contain terms oscillatory in space, so that the flow is oscillatory for a certain range of the suction parameter $V_1^2 < V^2 < \infty$, but only real exponential terms for the remaining values $0 < V^2 < V_1^2$ provided that A is negative; otherwise, that is if A is positive, the flow is oscillatory for $0 < V^2 < V_1^2$ and is non-oscillatory for the remaining values $V_1^2 < V^2 < \infty$.

Case 2, y is large.

When y is large, eq. (3.6) reduces to

$$k^3 + \frac{a}{v_0\beta}\left(\frac{\nu}{a}+\beta\right)k^2 - \frac{1}{\beta}k - \frac{a}{v_0\beta} = 0, \tag{3.9}$$

the roots of which are real and distinct or include a complex pair according as $\bar{\Delta}$ is negative or positive respectively, where

$$108\,\bar{\Delta} = -\left[\frac{4}{\beta^3} + V^2\left(\frac{B^2}{\beta^2} + 18\frac{B}{\beta} - 27\right) + 4V^4B^3\right], \tag{3.10}$$

$$V^2 = \left(\frac{a}{v_0\beta}\right)^2$$

and $B = \frac{\nu}{a} + \beta$.

Now $\bar{\Delta} \lessgtr 0$, according as $V^2 \lessgtr V_2^2$ where V_2^2 is positive root of the quadratic equation obtained by setting eq. (3.10) equal to zero, provided that B is negative; otherwise, that is if B is positive, $\bar{\Delta} \lessgtr 0$ according as $V^2 \lessgtr V_2^2$.

Thus, for large y, the flow will be oscillatory for a certain range of values of the suction parameter $V_2^2 < V^2 < \infty$ and non-oscillatory for remaining values $0 < V^2 < V_2^2$ provided that B is negative; otherwise, that is if B is positive, the flow will be oscillatory for $0 < V^2 < V_2^2$ and non-oscillatory for remaining values $V_2^2 < V^2 < \infty$.

The velocity components are

y small: $\quad u = ax\,e^{-k_1 y}$

$$v = -\left(\frac{a}{k_1}\right)(1 - e^{-k_1 y}) - v_0 \tag{3.11}$$

where k_1 is a root of eq. (3.7) with positive real part.

y large:

$$u=ax\, e^{-k_2 y}$$

$$v=\frac{a}{k_2}e^{-k_2 y}-c \tag{3.12}$$

where $c=v_0+\frac{a}{k_1}$ and k_2 is a root of eq. (3.9) with positive real part.

4. Skin friction

Shear stress at a point in the plate is

$$\tau_0=-\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}=k_1\mu u_0 \tag{4.1}$$

where $u_0=ax$. Dimensionless shear stress at a point on the plate is

$$\frac{\tau_0}{\rho u_0^2}=\frac{k_1\nu}{u_0}=k_1(\nu/a)^{\frac{1}{2}}R_x^{-\frac{1}{2}} \tag{4.2}$$

where $R_x=\frac{u_0 x}{\nu}$ is the Reynolds number at the point. The total drag on both sides of the plate upto the distance x from the slit is

$$bx\,\rho u_0^2 k_1(\nu/a)^{\frac{1}{2}}R_x^{-\frac{1}{2}} \tag{4.3}$$

where b is the width of the plate.

5. The boundary layer thickness

The boundary layer thickness δ is the value of y at which

$$v=-0.99c. \tag{5.1}$$

Using eq. (3.12), one gets

$$\delta=\frac{1}{k_2}\ \ln\frac{100k_1 a}{k_2(a+v_0k_1)}. \tag{5.2}$$

The dimensionless boundary layer thickness is

$$\frac{\delta}{l}=\left(\frac{a+v_0k_1}{\nu k_1k_2}\right)R_l^{-1}\ \ln\frac{100\ k_1 a}{k_2(a+v_0k_1)} \tag{5.3}$$

where $R_l=\frac{cl}{\nu}$ is the Reynolds number and l is the characteristic length.

6. Heat conduction in linear stretching case

The boundary layer equation governing the flow of heat is

$$u\ \frac{\partial T}{\partial x}+v\ \frac{\partial T}{\partial y}=\frac{\nu}{P}\ \frac{\partial^2 T}{\partial y^2} \tag{6.1}$$

where T is the temperature and P is the Prandtl number. Let the temperature of the sheet have the constant value T_p and the surrounding fluid the constant temperature T_s and put

$$\overline{T}=\frac{T-T_s}{T_p-T_s},\quad \eta=ky. \tag{6.2}$$

If the heat conduction is in the y direction only, then eq. (6.1) reduces to

$$v\frac{d\overline{T}}{d\eta}=\frac{\nu k}{P}\frac{d^2\overline{T}}{d\eta^2} \tag{6.3}$$

The boundary conditions are

$$\begin{aligned}&\eta=0\ :\ \overline{T}=1\\&\eta=\infty:\ \overline{T}=0\end{aligned} \tag{6.4}$$

For small y, $v=-\frac{a}{k_1}(1-e^{-\eta})-v_0$ where $\eta=k_1y$.

Therefore, the solution of e.q (6.3) subject to eq. (6.4) is

$$\overline{T}=\frac{\Gamma\left[\frac{Pa}{\nu k_1^2}\left(1+\frac{k_1v_0}{a}\right),\ \frac{Pa}{\nu k_1^2}e^{-\eta}\right]}{\Gamma\left[\frac{Pa}{\nu k_1^2}\left(1+\frac{k_1v_0}{a}\right),\ \frac{Pa}{\nu k_1^2}\right]} \tag{6.5}$$

where $\Gamma(n,x)$ is the incomplete gamma function $\int_0^x t^{n-1}\,e^{-t}\,dt$, $\mathrm{Re}(n)>0$.

For large y, $v=-\frac{a}{k_2}\left(\frac{k_2c}{a}-e^{-\eta}\right)$ where $\eta=k_2y$.

Therefore, the solution of eq. (6.3) subject to eq. (6.4) is

$$\overline{T}=\frac{\Gamma\left[\frac{Pc}{\nu k_2},\ \frac{Pa}{\nu k_2^2}e^{-\eta}\right]}{\Gamma\left[\frac{Pc}{\nu k_2},\ \frac{Pa}{\nu k_2^2}\right]}. \tag{6.6}$$

Heat flux from one side of the plate is

$$q(x)=K\left(\frac{\partial T}{\partial y}\right)_{y=0}=\frac{K}{x}R_x^{\frac{1}{2}}k_1(\nu/a)^{\frac{1}{2}}(T_p-T_s)\ M(aP/\nu k_1^2), \tag{6.7}$$

where K is the coefficient of conductivity and

$$M(m)=\frac{m^{m(1+k_1v_0/a)}e^{-m}}{\Gamma[m(1+k_1v_0/a),\ m]},\quad m=\frac{ap}{\nu k_1^2}. \tag{6.8}$$

7. Discussion

For zero-suction case ($v_0=0$), $k_1=(a/\nu k')^{\frac{1}{2}}$ and $k_2=(a/\nu k'')^{\frac{1}{2}}$, where $k'=1+(5\beta+2\gamma)a/\nu$ and $k''=1+\beta a/\nu$. Then eqs. (4.2), (5.3) and (6.7) reduce to

$$\frac{\tau_0}{\rho u_0^2}=k'^{-\frac{1}{2}} R_x^{-\frac{1}{2}}, \tag{7.1}$$

$$\frac{\delta}{l}=R_l^{-1}(k'k'')^{\frac{1}{2}} \ln 100 \, (k''/k')^{\frac{1}{2}}, \tag{7.2}$$

$$\text{and } q(x)=\frac{K}{x}(T_p-T_s) \, R_x^{\frac{1}{2}} \, k'^{-\frac{1}{2}} H(Pk'), \tag{7.3}$$

where $H(m)=\frac{m^m e^{-m}}{\Gamma(m, m)}$, $m=Pk'$, respectively, which agree with the results obtained by Siddappa and Khapate [3]. Hence, due to suction v_0, the skin friction is reduced by a factor $k_1(k'\nu/a)^{\frac{1}{2}}$, the boundary layer thickness is reduced by a factor

$$\frac{[(a+v_0k_1)/\nu k_1k_2] \ \ln \ [100 \ k_1a/k_2(a+v_0k_1)]}{(k'k'')^{\frac{1}{2}} \ \ln \ [100(k''/k')^{\frac{1}{2}}]}$$

and the heat flux is reduced by a factor $[k_1(\nu/ak')^{\frac{1}{2}} M(aP/\nu k_1^2)]/H(Pk')$.

It is striking to note that the above analysis of the nature of flow holds good for negative values of v_0 also (i.e. injection case), since the nature of flow depends on the parameter $V^2=(a/v_0\beta)^2$.

Department of Mathematics
Karnatak University Post-Graduate
Centre, Gulbarga,
Karnataka, India.

Department of Mathematics,
Karnataka Regional Engineering College,
P. O. Srinivasnagar-574157,
(South Kanara District)

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