

A NOTE ON POSITIVITY OF INTEGRALS AND ENVELOPING SERIES

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1. Introduction

In recent years various methods have been used for showing positivity of integrals involving special functions, especially for Bessel functions, see for example Gasper [2] and Steinig [8].

The intention of this note is to first obtain an elementary proof of generalisation of Bernoulli's inequality for the binomial theorem, which is originally due to Gerber [3], and then to use this simple result in obtaining positivity of certain integrals involving special functions, which then can also be used to derive enveloping series for some of these functions, see Pólya and Szegő [5].

Recall that for a Maclaurin series expansion of the function f of the form

$$f(x) = \sum_{\nu=0}^n \frac{f^{(\nu)}(0)x^\nu}{\nu!} + R_n(x),$$

where the remainder is $R_n(x) = \sum_{\nu=n+1}^{\infty} \frac{f^{(\nu)}(0)x^\nu}{\nu!}$ can be expressed in terms of an integral

$$R_n(x) = \frac{1}{n!} \int_0^x (x-t)^n D^{n+1}[f(t)] dt,$$

or in Cauchy's form as

$$R_n(x) = (1-\theta_n)^n \frac{f^{(n+1)}(\theta_n x)}{n!} x^{n+1} \text{ for some } \theta_n \text{ in } 0 < \theta_n < 1.$$

Here the symbol D^k or $f^{(k)}$ is used to denote k^{th} derivative.

2. Gerber's generalisation of bernoulli's inequality

Let $f(x) = (1+x)^\gamma$ for $x > -1$ and any $\gamma \in R$. Then $f(x) = \sum_{\nu=0}^n \binom{\gamma}{\nu} x^\nu + R_n(x)$

where $R_n(x) = \binom{\gamma}{n+1} \frac{(1-\theta_n)^n}{(1+\theta_n x)^{n+1-\gamma}} x^{n+1}$ for some θ_n in $0 < \theta_n < 1$.

Here $\binom{\lambda}{\nu}$ are generalised binomial coefficients defined by

$$\binom{\lambda}{\nu} = \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-\nu+1)}{\nu!}, \text{ for } \nu \in \mathbf{N}, \lambda \in \mathbf{R}.$$

Hence

$$\binom{\gamma}{n+1} x^{n+1} R_n(x) = \left[\binom{\gamma}{n+1} x^{n+1} \right]^2 \frac{(1-\theta_n)^n}{(1+\theta_n x)^{n+1-r}} \geq 0$$

and $R_n(x) = 0$ if $x=0$ or $\gamma=0, 1, \dots, n$.

That is,

$$\binom{\gamma}{n+1} x^{n+1} \left[(1+x)^r - \sum_{\nu=0}^n \binom{\gamma}{\nu} x^\nu \right] \geq 0 \quad (1)$$

for $x > -1$ and $\gamma \in \mathbf{R}$ with the term in square brackets equal to zero only when $x=0$ or $\gamma=0, 1, \dots, n$ provided that $0 \neq x > -1$.

An alternative proof of this is recently given by Ross [6].

Similarly,

$$\frac{x^{n+1}}{(n+1)!} \left[e^x - \sum_{\nu=0}^n \frac{x^\nu}{\nu!} \right] \geq 0, \quad x \in \mathbf{R}, \quad n \in \mathbf{N}.$$

This is a trivial result in case $x \geq 0$ but not quite so simple if $x < 0$ although it has been known for many years, see Pólya and Szegő [5].

Now, we prove the following:

3. Theorem

The integral

$$\int_0^x (x-t)^{m+s} (1-t)^{r-m-s-1} {}_2F_1(-s, \gamma+1; m+2; t) dt \geq 0, \quad (2)$$

for $0 \leq x < 1$, $s, m \in \mathbf{N}$, $\gamma > m+s$.

PROOF. From the inequality (1) we obtain

$$(-1)^{m+1} \left[x^s (1-x)^r - \sum_{\nu=0}^m \binom{\gamma}{\nu} (-1)^\nu x^{\nu+s} \right] \geq 0$$

for some $s \in \mathbf{N}$, $r > 0$, $0 \leq x < 1$.

Since the term in square brackets is the remainder after $(m+s+1)$ terms in the Maclaurin series expansion of $x^s(1-x)^r$ it follows that

$$(-1)^{m+1} \int_0^x (x-t)^{m+s} D^{m+s+1} [t^s (1-t)^r] dt \geq 0.$$

Now, according to Luke [4, p. 258]

$$D^r [x^s (1-x)^r] = \frac{r!(-\gamma)_{r-s}}{(r-s)!} (1-x)^{r-r} 2^F 1(-s, \gamma+1; r-s+1; x); r, s \in N, s \leq r.$$

Thus,

$$(-1)^{m+1} \int_0^x (x-t)^{m+s} (-\gamma)_{m+1} (1-t)^{r-m-s-1} 2^F 1(-s, \gamma+1; m+2; t) dt \geq 0$$

and since $(-\gamma)_{m+1} = (-1)^{m+1} \gamma(\gamma-1)\dots(\gamma-m)$, the above result (2) follows.

Now the Jacobi polynomials may be defined by the relation

$$P_n^{(\alpha, \beta)}(x) = \binom{n+\alpha}{n} 2^F 1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right).$$

Then, on using the above theorem we obtain

$$\int_0^1 (1-t)^{m+n} (1-xt)^\beta P_n^{(m+1, \beta)}(1-2xt) dt \geq 0 \tag{3}$$

for $0 \leq x < 1, \beta > -1, m, n \in N$.

This appears to be a new inequality from which positivity of several integrals can be obtained. In fact, a better inequality than above can be obtained with less restricted conditions. To see this, let

$$S_{\zeta, \alpha, \beta} = \int_0^x (x-t)^\zeta (1-t)^\beta P_n^{(\alpha, \beta)}(1-2t) dt, \quad 0 \leq x < 1, \zeta, \beta, \alpha > -1.$$

Then, on using the Kummer's transformation

$$2^F 1(a, b; c; x) = (1-x)^{c-a-b} 2^F 1(c-a, c-b; c; x)$$

and integration by parts, we obtain

$$\begin{aligned} \binom{n+\alpha}{n}^{-1} S_{\zeta, \alpha, \beta} &= \int_0^x (x-t)^\zeta 2^F 1(\alpha+n+1, -(n+\beta); \alpha+1; t) dt \\ &= \frac{1}{(\zeta+1)} x^{\zeta+1} - \frac{1}{(\zeta+1)} \frac{(\alpha+n+1)(n+\beta)}{(\alpha+1)} S_{\zeta+1, \alpha+1, \beta-1} \end{aligned}$$

or,

$$\frac{(\alpha+n+1)(n+\beta)}{(\alpha+1)} S_{\zeta+1, \alpha+1, \beta-1} = x^{\zeta+1} - \binom{n+\alpha}{n}^{-1} (\zeta+1) S_{\zeta, \alpha, \beta}.$$

But since, Szegö [9]

$$|P_n^{(\alpha, \beta)}(1-2x)| \leq \binom{n+\alpha}{n}, \quad 0 \leq x \leq 1, \alpha \geq \beta > -1, \alpha \geq -\frac{1}{2}.$$

and since

$$(1-x)^\beta \leq 1 \text{ for } \beta > 0, 0 \leq x < 1,$$

it appears that

$$|S_{\zeta, \alpha, \beta}| \leq \binom{n+\alpha}{n} \cdot \frac{1}{(\zeta+1)} x^{\zeta+1} \text{ for } 0 \leq x < 1, \alpha \geq \beta > 0, \zeta > -1,$$

hence

$$S_{\zeta+1, \alpha+1, \beta-1} \geq 0 \text{ for } 0 \leq x < 1, \alpha \geq \beta > 0, \zeta > -1.$$

In other words,

$$\int_0^x (x-t)^r (1-t)^\eta P_n^{(\mu, \eta)}(1-2t) dt \geq 0; 0 \leq x < 1, \mu \geq \eta + 2 > 1, r > 0.$$

Now, on using the known identity, Szegö [9]

$$\lim_{\beta \rightarrow \infty} P_n^{(\alpha, \beta)}\left(1 - \frac{2x}{\beta}\right) = L_n^{(\alpha)}(x)$$

where $L_n^{(\alpha)}(x)$ are usual Laguerre polynomials, and the observation that the limit of integral equals integral of the limit, it follows from (3) that

$$\int_0^1 (1-t)^{m+n} e^{-xt} L_n^{(m+1)}(xt) dt \geq 0, m, n \in \mathbb{N}, x \geq 0.$$

Once again, on using integration by parts, it appears that

$$\int_0^x (x-t)^r e^{-t} L_n^{(\eta)}(t) dt \geq 0, \text{ for } r > 0, \eta \geq \frac{1}{2}, n \in \mathbb{N}, x \geq 0 \quad (4)$$

which is a better result than one that appears in Ross and Mahajan [7]. Now on using the relation between the generalised Hermite and Laguerre polynomials, Szegö [9], we obtain

$$(-1)^n \int_0^x (x-t)^r e^{-t} H_{2n}^{(k)}(\sqrt{t}) dt \geq 0$$

for all $x \geq 0, r > 0, k \geq 1, n \in \mathbb{N}$ and

$$(-1)^n \int_0^x (x-t)^r e^{-t} t^{-\frac{1}{2}} H_{2n+1}^{(k)}(\sqrt{t}) dt \geq 0$$

for all $x \geq 0, r > 0, k \geq 0, n \in \mathbb{N}$.

Also since, Szegö [9],

$$\lim_{n \rightarrow \infty} \left[e^{-x/n} \frac{1}{n^\eta} L_n^{(\eta)}(x/n) \right] = \lim_{n \rightarrow \infty} \frac{1}{n^\eta} L_n^{(\eta)}(x/n) = x^{-\eta/2} J_\eta(2\sqrt{x})$$

where $J_\eta(x)$ are the Bessel functions of the first kind, we obtain from (4) the inequality

$$\int_0^x (x-t)^r t^{-\eta/2} J_\eta(2\sqrt{t}) dt \geq 0, \text{ for all } x \geq 0, r > 0, \eta \geq \frac{1}{2}.$$

This last inequality together with the simple transformation

$$y = 2\sqrt{t}, \eta = \frac{\mu - \nu + 1}{2}, \gamma = \frac{\mu + \nu - 1}{2},$$

and the symmetry relation for the Lommel function $s_{\mu, \nu}(x) = s_{\mu, -\nu}(x)$ implies that $s_{\mu, \nu}(x) \geq 0$ for $x > 0$ and provided $\mu + \nu > \frac{1}{2}$, $|\nu| \leq \mu$.

This result in a slightly weaker form is also obtained in Steinig [8] who employs an oscillation theorem of E. Makai and fractional integrations.

4. Enveloping series

The results of previous section may also be used to derive enveloping series for some of these special functions. For example, since

$$D[e^{-t} L_n^{(\alpha)}(t)] = -e^{-t} L_n^{(\alpha+1)}(t)$$

it follows from inequality (4) that

$$(-1)^{m+1} \int_0^x (x-t)^m D^{m+1}[e^{-t} L_n^{(\alpha)}(t)] dt \geq 0$$

for $x \geq 0$, $m = 1, 2, \dots$, $\alpha > -1$. In case $m = 0$ the inequality amounts to

$$e^{-x} L_n^{(\alpha)}(x) \leq \frac{(\alpha+1)_n}{n!} \text{ which holds for } \alpha \geq -\frac{1}{2}. \text{ This implies that}$$

$$(-1)^{m+1} \left[e^{-x} L_n^{(\alpha)}(x) - \sum_{\nu=0}^m \binom{n+\nu+\alpha}{n} \frac{(-x)^\nu}{\nu!} \right] \geq 0$$

for all $x \geq 0$, $m \in \mathbb{N}$, $\alpha \geq -\frac{1}{2}$. In case $m \neq 0$, the inequality holds for $\alpha > -1$.

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