Kyungpook Math. J.
Volume 20, Number 2
December 1980.

SOME REMARKS ON θ-RIGIDITY

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Let X be a topological space and let $A\subset X$. We will denote by $\mathrm{cl}(A)$ and $\Sigma(A)$ the closure of A and the family of open subsets of X which contain A, respectively. Veličko [V] has defined the *θ-closure* of A ($\mathrm{cl}_\theta(A)$) to be $\{x\in X$: each $V\in\Sigma(x)$ satisfies $A\cap \mathrm{cl}(V)\neq\phi\}$; A is *θ-closed* if $\mathrm{cl}_\theta(A)=A$. If Ω is a filterbase on X, the *θ-adherence of* Ω ($\mathrm{ad}_\theta\Omega$) is $\bigcap_{\Omega}\mathrm{cl}_\theta(F)$. It is known [J] that $\mathrm{cl}_\theta(A)=\bigcap_{\Sigma(A)}\mathrm{cl}(V)$ and, consequently, that $\mathrm{ad}_\theta\Omega=\mathrm{ad}\bigcup_{\Omega}\Sigma(F)$ for any filterbase Ω on X. A is *θ-rigid in* X [DP_1] if each filterbase Ω on X satisfying $F\cap \mathrm{cl}(V)\neq\phi$ for all $F\in\Omega$ and $V\in\Sigma(A)$ also satisfies $A\cap \mathrm{ad}_\theta\Omega\neq\phi$. Dickman and Porter [$\mathrm{DP}_1$] have found θ-rigid subsets to be useful in the study of the extension function problem for θ-continuous functions between Hausdorff spaces. A is *quasi H-closed* [***QHC***] *relative to* X if each filterbase Ω on A satisfies $A\cap\mathrm{ad}_\theta\Omega\neq\phi$ [H]. If X is QHC relative to X we say that X is an *$H(i)$ space* [PT]. A Hausdorff $H(i)$ space is an *H-closed space* and a QHC relative to X subset is called an *H-set* if X is Hausdorff [V]. The author has shown [J] that $\mathrm{cl}_\theta(A)$ is QHC relative to X if X is an $H(i)$ space. It is known that a θ-rigid subset of any space is QHC relative to the space and that a θ-rigid subset of a Hausdorff space is θ-closed [DP_2].

In this paper we establish that a θ-closed subset of an $H(i)$ space is θ-rigid and, consequently, that the family of θ-rigid subsets of an H-closed space coincides with the family of θ-closed subsets. As a consequence of this realization, we are able to improve a number of known results on subsets of H-closed spaces, to offer a characterization, in terms of θ-rigid subsets of various spaces, of those Hausdorff spaces in which the Fomin H-closed extension operator commutes with the projective cover (absolute) operator, and to offer some new characterizations of locally H-closed spaces. We also present a product theorem for θ-rigid subsets.

In our first result we give some characterizations of θ-rigid subsets which will be used in the sequel. We recall that a filterbase Ω on a space X *θ-converges to* $x\in X$ ($\Omega\to_\theta x$) if for each $V\in\Sigma(x)$ there is an $F\in\Omega$ satisfying

$F\subset \mathrm{cl}(V)$ [V].

PROPOSITION 1. *The following statements are equivalent for a space X and $A\subset X$:*

(a) *A is θ-rigid in X.*

(b) *Each open filterbase Ω on X satisfying $V\cap W\neq\phi$ for all $V\in\Omega$ and $W\in \Sigma(A)$ also satisfies $A\cap ad\Omega\neq\phi$.*

(c) *Each filterbase Ω on X satisfying $V\cap W\neq\phi$ for all $V\in\bigcup_{\Omega}\Sigma(F)$ and $W\in \Sigma(A)$ also satisfies $A\cap ad_{\theta}\Omega\neq\phi$.*

(d) *Each base, $\mathscr{U}$, for an ultrafilter on X satisfying $F\cap cl(W)\neq\phi$ for all $F\in\mathscr{U}$ and $W\in\Sigma(A)$ θ-converges to some point in A.*

PROOF. It is obvious that (a) implies (b); that (b) implies (c) follows easily from the remarks in paragraph 1. Under the hypothesis of (d), we see that all $V\in\bigcup_{\mathscr{U}}\Sigma(F)$ and $W\in\Sigma(A)$ satisfy $V\cap \mathrm{cl}(W)\neq\phi$ and, consequently, $V\cap W\neq\phi$. Hence, assuming (c), $A\cap \mathrm{ad}_{\theta}\mathscr{U}\neq\phi$. Since $\mathscr{U}$ is a base for an ultrafilter on X, it follows that $\mathscr{U}\rightarrow_{\theta}x$ for each $x\in A\cap \mathrm{ad}_{\theta}\mathscr{U}$ and (c) implies (d). Now assume (d), and let Ω be a filterbase on X such that all $F\in\Omega$ and $W\in\Sigma(A)$ satisfy $F\cap \mathrm{cl}(W)\neq\phi$. Let $\mathscr{U}$ be an ultrafilter on X containing $\Omega\cup\{\mathrm{cl}(W): W\in\Sigma(A)\}$. Then all $B\in\mathscr{U}$ and $W\in\Sigma(A)$ satisfy $B\cap \mathrm{cl}(W)\neq\phi$. Hence $\mathscr{U}\rightarrow_{\theta}x$ for some $x\in A$. Since $x\in A\cap \mathrm{ad}_{\theta}\Omega$ we conclude that A is θ-rigid and that (d) implies (a). The proof is complete.

Our next result is a product theorem for θ-rigid subsets. If $\{X_{\alpha}: \alpha\in\Delta\}$ is a family of sets we denote the product of these sets by $\prod_{\Delta}X_{\alpha}$ and, for $\alpha\in\Delta$, we denote the projection of $\prod_{\Delta}X_{\alpha}$ onto X_{α} by π_{α}.

THEOREM 2. *Let $\{X_{\alpha}: \alpha\in\Delta\}$ be a family of spaces and, for each $\alpha\in\Delta$, let A_{α} be a nonempty subset of X_{α}. A necessary and sufficient condition for $\prod A_{\alpha}$ to be θ-rigid in $\prod_{\Delta}X_{\alpha}$ is that A_{α} be θ-rigid in X_{α} for each $\alpha\in\Delta$.*

PROOF. The necessity of the condition follows from the readily established fact that an open continuous image of a θ-rigid subset is θ-rigid. For the proof of the sufficiency, let $\mathscr{U}$ be a base for an ultrafilter on $\prod_{\Delta}X_{\alpha}$ satisfying $B\cap \mathrm{cl}(W)\neq\phi$ for all $B\in\mathscr{U}$ and $W\in\Sigma(\prod A_{\alpha})$. Then, for $\alpha\in\Delta$, $\pi_{\alpha}(\mathscr{U})$ is a base for an ultrafilter on X_α. If $V\in\Sigma(A_\alpha)$, then $\pi_\alpha^{-1}(V)\in\Sigma(\prod_\Delta A_\alpha)$ and, therefore, any $B\in\mathscr{U}$ satisfies $B\cap\pi_\alpha^{-1}(\mathrm{cl}(V))=B\cap\mathrm{cl}(\pi_\alpha^{-1}(V))\neq\phi$. Hence $\pi_\alpha(B)\cap\mathrm{cl}(V)\neq\phi$ is satisfied for all $B\in\mathscr{U}$ and $V\in\Sigma(A_\alpha)$. Consequently, from Proposition 1 (d), there is an $x_\alpha\in A_\alpha$ such that $\pi_\alpha(\mathscr{U})\rightarrow_\theta x_\alpha$. Let $x\in\prod_\Delta X_\alpha$ with $\pi_\alpha(x)=x_\alpha$ for all $\alpha\in\Delta$. Then $x\in\prod_\Delta A_\alpha$ and $\mathscr{U}\rightarrow_\theta x$. The proof is complete.

The following theorem improves a number of known results and is used extensively in the remainder of this paper.

THEOREM 3. *A θ-closed subset of an $H(i)$ space is θ-rigid in the space.*

PROOF. Let Ω be an open filterbase on the $H(i)$ space X, let A be θ-closed in X and suppose that $V\cap W\neq\phi$ is satisfied for all $V\in\Omega$ and $W\in\Sigma(A)$. Then $\Omega_1=\{V\cap W : V\in\Omega,\ W\in\Sigma(A)\}$ is an open filterbase on X. Hence $\phi\neq\mathrm{ad}\Omega_1\subset\mathrm{cl}_\theta(A)\cap\mathrm{ad}\Omega=A\cap\mathrm{ad}\Omega$. Therefore, by Proposition 1 (b), A is θ-rigid. The proof is complete.

COROLLARY 4. *A subset of an H-closed space X is θ-rigid in X if and only if it is θ-closed in X.*

COROLLARY 5. [J]. *A θ-closed subset of an $H(i)$ space is QHC relative to the space.*

COROLLARY 6. [V]. *A θ-closed subset of an H-closed space is an H-set.*

Before moving to other results in this paper, we need some additional definitions and terminology. An *open filter on a space X* is a nonempty collection of open sets Ω satisfying the following properties: (1) $\phi\notin\Omega$, (2) If V, $W\in\Omega$, then $V\cap W\in\Omega$, and (3) If $V\in\Omega$ and W is open in X with $V\subset W$, then $W\in\Omega$. An *open ultrafilter* is an open filter which is maximal in the collection of open filters. Let X be a Hausdorff space and let $X^*=X\cup\{\mathscr{U} : \mathscr{U}$ is a free open ultrafilter on $X\}$. For each open V of X, let $0(V)=V\cup\{\mathscr{U}\in X^*-X : V\in\mathscr{U}\}$. Then $\{0(V) : V$ open in $X\}$ is an open base for a topology on X^*. X^* with this topology is an H-closed extension of X [F] called the *Fomin extension of X* and denoted by σX ; X^* with the topology generated by the open base $\{V : V$ open in $X\}\cup\{V\cup\{\mathscr{U}\} : V\in\mathscr{U},\ \mathscr{U}\in X^*-X\}$ is an H-closed extension of X [K] called the *Katětov extension of X* and denoted by κX.

THEOREM 7. *Let X be a Hausdorff space. The following statements are equivalent for $A \subset X$:*

(a) *A is θ-rigid in κX.*

(b) *A is θ-rigid in X.*

(c) *A is θ-rigid in σX.*

(d) *A is θ-rigid in some H-closed extension of X.*

PROOF. The proof follows easily from Corollary 4 above, (2.2) from [DP_2], and (3.2) from [DP_2].

Now, for a Hausdorff space X, let θX denote $\{\mathscr{U} : \mathscr{U}$ is an open ultrafilter on $X\}$. For each open V in X, let $0(V)$ denote $\{\mathscr{U} \in \theta X : V \in \mathscr{U}\}$; $\{0(V) : V$ open in $X\}$ is a base for an extremally disconnected, compact Hausdorff topology on θX [IF]. By Theorem 5.2 in [PV] there is a θ-continous, perfect irreducible function $\pi : \theta X \to \sigma X$ defined by $\pi(\mathscr{U}) = \mathscr{U}$ for each free open ultrafilter $\mathscr{U}$ on X and $\pi(\mathscr{U}) = x$ where x is the unique convergent point of the fixed open ultrafilter $\mathscr{U}$. It is established in [DP_2] that if X is a Hausdorff space and $A \subset X$, then $\pi^{-1}(A)$ is compact if and only if A is θ-closed in κX. In view of this result and Corollary 4 above, the following theorem follows.

THEOREM 8. *If X is a Hausdorff space and $A \subset X$, then $\pi^{-1}(A)$ is compact if and only if A is θ-rigid in κX.*

For a Hausdorff space X, the subspace $\{\mathscr{U} \in \theta X : \mathscr{U}$ is fixed$\}$ of θX is denoted by EX and is called the ***absolute of*** X. Using Corollary 4 above and Corollary (3.5) of [DP_2] we obtain the following characterization of those Hausdorff spaces in which the Fomin H-closed extension operator commutes with the absolute operator.

THEOREM 9. *Let X be a Hausdorff space. Then $\sigma(EX) = E(\sigma X)$ if and only if the set of nonisolated points of X is θ-rigid in κX.*

A Hausdorff space X s *locally H-closed* if each point in X has an H-closed neighborhood [O]. Properties of locally H-closed spaces have been studied in [P]. A number of characterizations appear in [P], [PV]. In our next theorem we utilize Corollary 4 above to offer two new characterizations.

THEOREM 10. *The following statements are equivalent for a Hausdorff space X:*

(a) *X is locally H-closed.*

(b) *κX-X is θ-closed in κX.*

(c) *κX-X is θ-rigid in κX.*

PROOF. The equivalence of (b) and (c) follows directly from Corollary 4. To see that (a) implies (b), let $x\in X$ and let H be an H-closed neighborhood of x in X. If $\mathscr{U}$ is a free open ultrafilter on X there is a $W\in\mathscr{U}$ satisfying $H\cap W=\phi$. Otherwise $\mathrm{ad}\mathscr{U}\neq\phi$. Hence $\mathrm{cl}_{\kappa X}(H)=H$ and (b) holds. Now assume (b) and let $x\in X$. Then, since κX-X is θ-closed in κX, there is a $V\in\Sigma(x)$ in X such that $\mathrm{cl}_{\kappa X}(V)\cap(\kappa X\text{-}X)=\phi$. Hence $\mathrm{cl}_{\kappa X}(V)=\mathrm{cl}(V)$. Let Ω be a family of open subsets of X such that $\Omega_1=\{F\cap\mathrm{cl}(V): F\in\Omega\}$ is an open filterbase on $\mathrm{cl}(V)$. Since $\mathrm{cl}_{\kappa X}(V)$ is H-closed and Ω_1 is an open filterbase on $\mathrm{cl}_{\kappa X}(V)$, we have $\phi\neq\bigcap_{\Omega}\mathrm{cl}_{\kappa X}(F\cap\mathrm{cl}_{\kappa X}(V))=\bigcap_{\Omega}\mathrm{cl}_{\kappa X}(F\cap V)=\bigcap_{\Omega}\mathrm{cl}(F\cap V)=\bigcap_{\Omega}\mathrm{cl}(F\cap\mathrm{cl}(V))$. Therefore $\mathrm{cl}(V)$ is an H-closed subset of X, X is locally H-closed, (b) implies (a) and the proof is complete.

Finally, we note that the classic example of a minimal Hausdorff non-compact space [B] has an element x and an open set V satisfying $x\in\mathrm{cl}_\theta(\mathrm{cl}(V))$-$\mathrm{cl}(V)$. So $\mathrm{cl}(V)$ is not θ-closed and, consequently, not θ-rigid. Hence $\mathrm{cl}_\theta(A)$ could fail to be θ-rigid for a subset A of an H-closed space.

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