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HARMONIC PSEUDOMETRIC AND THE PRODUCT PROPERY OF PSEUDOMETRIC

By Sang Moon Kim

Introduction

Let S be a complex manifold, and let ρ be non negative real valued function on $S\times S$. If ρ satisfies $\rho(x,y)=\rho(y,x)$, $\rho(x,y)+\rho(y,z)\geq\rho(x,z)$ and $\rho(x,x)=0$, then ρ is called as a *pseudometric* on S. We call any system which assigns a pseudometric to each complex manifold a *Schwarz-Pick system* if it satisfies the following conditon:

(1) the pseudometric assigned to the unit open disk is the Poincaré metric,
(2) if ρ_1 and ρ_2 are the pseudometrics assigned to m nif olds S_1 and S_2 respectively, then $\rho_2(h(x), h(y))\leq\rho_1(x,y)$ for all holomorphic mapping $h: S_1\to S_2$ and for any pair of points x and y in S_1, that is the pseudometric has the distance decreasing property respect to the holomorphic map.

Now let ρ_i and ρ be the pseudometrics assigned to D_i and $D_1\times D_2$ respectively, then it satisfies (3) $\max\{\rho_1(x_1,y_1), \rho_2(x_2,y_2)\}\leq\rho((x_1,y_1),(x_2,y_2))$ for all x_1, x_2 in D_1 and y_1, y_2 in D_2.

The Kobayashi and the Caratheodory pseudometrics are the two extremal pseudometrics given by Scwarz-Pick system. The Caratheodory pseudometric is the smallest and the Kobayashi pseudometric is the largest one which can be assigned to complex manifolds.

Let P be the Poincaré metric defined on the unit open disk D in the complex plane and let S be a complex manifold then we consider the real valued function $C: S\times S\to R$ such that, $C(x,y)=\sup\{P(g(x), g(y))\mid g\in G\}$ where G is the set of all holomorphic functions $g: S\to D$. The function C is called the *Caratheodory Pseudometric* on S.

We define the Kobayashi pseudometric K on a complex manifold as follows. Given two points x and y in S, we choose points $x=x_0, x_1, \cdots, x_{k-1}, x_k=y$ of S, points $a_1,\cdots,a_k$, $b_1,\cdots,b_k$ of D, and f_k holomorphic mappings $f_1,\cdots,f_k$ of D into S such that $f_i(a_i)=x_{i-1}$ and $f_i(b_i)=x_i$ for $i=1, 2,\cdots, k$. For each choice of points and mappings thus made, we consider the numbers $P(a_1, b_1)+, \cdots, +P(a_k, b_k)$.

Let $K(x, y)$ be the infimum of the numbers obtained in this manner for all possible choices. The function K is called the ***Kobayashi Pseudometric*** for S.

One calls ***a pseudometric has the product property*** if the equality holds in the equation (3). It is known that the Kobayashi pseudometric has the product property, see [4].

In the following we construct some pseudometics and show that in some sense, almost all of the pseudometric does not have the product property. In the following pseudometric means always the pseudometric assigned by a Scwarz-Pick system. It is easy to see that the Kobayashi and the Caratheodory pseudometrics assigned by the Schwarz-Pick systems.

Construction of pseudometrics and it's product property:

Let α and β be given pseudometrics on S, then we define a family of new pseudometrics γ_t by $\gamma_t(x, y)=t\alpha(x, y)+(1-t)\beta(x, y)$, $x, y\in D$ and $0\leq t\leq 1$.

Then it is clear that the γ_t is a pseudometric for each fixed t.

We want to define a new pseudometric H using harmonic functions on complex manifolds S. We call a real valued function u is harmonic if it is locally the real part of a holomorphic function. That is for each $z\in S$, there is an holomorphic function f defined on an open neighborhood V of z with real $(f(z))=u(z)$.

Let D be the open unit disk in the complex plane and let G be the family of all harmonic functions h on D which satisfies $h(0)=0$ and $-1<h(z)<1$ for all $z\in D$. For $0<r<1$, we define $m(r)=\sup\{h(r) : h\in G\}$. To define the H, we need to know the value of $m(r)$. The value of $m(r)$ must be known but we have no convenient references, in the following we calculate $m(r)$ by the method suggested by Donald Sarason. Suppose L^1 and L^∞ be the set of all integrable and all bounded measurable functions on ∂D (=boundary of D). We know that every bounded harmonic function on D can be represented as a Poisson integral of a function in L^∞. With this setting we want to evoluate $m(r)$ by the duality of L^1 with L^∞. Since the bounded harmonic functions can be represented by elements in L^∞, in the following when $h\in L^\infty$ we also consider h as the harmonic function on D with the boundary value h. Let $M=\{h\in L^\infty : h(0)=0\}$, here $h(0)$ denotes $1/2-\pi\int_{-\pi}^{\pi} h(\theta)d\theta$, that is the value of harmonic function at 0 with boundary value h. Let C denote all constant functions on ∂D, then M can be considered as the space of continuous linear functionals on the quotient space L^1/C. As an element of the Banach space L^1/C, the poisson kernel $Pr(\theta)$ on D has norm $\|Pr(\theta)\|=\sup 1/2\pi\int_{-\pi}^{\pi} Pr(\theta)\ h(\theta)\ d\theta$, where h moves all the elements of the dual space $(L^1/C)^*=M$ with norm 1. We have the following identities $m(r)=\sup\{|(h)|: h\in M,\ \|h\|=1\}=\|Pr(\theta)\|=\inf_{c\in C}\|Pr(\theta)-c\|$

where $\|Pr(\theta)\|$ denotes the norm in L^1/C and $\|Pr(\theta)-c\|$ is the norm in L^1. To know the $m(r)$ we need to evoluate $\inf_{c\in C} 1/2\pi\int_{-\pi}^{\pi} |Pr(\theta)-c|d\theta$.

Let $C=Pa(t)=(1-r^2/1-2\ r\cos t+r^2)$, then by some calculation we get

$$(3)\ Pr(\theta)-c=1/2\pi\int_{-t}^{t}\frac{1-r^2}{1-2r\cos\theta+r^2}\,d\theta-1+\frac{1-r^2}{1-2r\cos t+r^2}\left(\frac{\pi-2t}{\pi}\right).$$

Taking derivative respect to t, we have that $|Pr(\theta)-c|$ has the smallest value at $t=\pi/2$ and hence $c=(1-r^2/1+r^2)$. Evaluating (3) with $t=\pi/2$ we have $m(r)=\frac{\pi}{4}\arctan r$.

Using this $m(r)$ we define the harmonic pseudometric H as the following.

DEFINITION. Let P be the Poincaré metric of the unit open disk D in the complex plane. Let S be a complex manifold, and set $m(x, y)=\sup\{h(y): h\in G\}$, where G denotes all of the harmonic function $h: S\to(-1, 1)$ with $h((x)=0$. Let $n(x, y)=\tan\frac{\pi}{4}\ m(x, y)$ and consider the real valued function $H: S\times S\to R$, with $H(x, y)=P(0, n(x, y))$, for any $(x, y)\in S\times S$. We call the H as the ***harmonic pseudometric*** on S.

We list here the properties of the pseudometric H as a proposition.

PROPOSITION 1. *The harmonic pseudometric H is given by a Schwarz-Pick system, and the pseudometic is different from the Kobayashi and the Caratheodory pseudometric.*

PROOF. By the construction it is clear that the pseudometric H satisfies all conditions of the Schwarz-Pick system. The harmonic pseudometric H is identically zero on any compact complex surface hence it differs from the Kobayashi pseudometric. There is an open set Ω in the complex plane which do not admits bounded non constant analytic function but it carries a non constant bounded harmonic function, see [1], on such a surface we know that H is bigger than the Caratheodory pseudometric. Hence we can say that the harmonic pseudometric is different from Caratheodory and the Kobayashi pseudometric. Thus we finished the proof of the proposition.

Now we want to show that many pseudometrics does not have the product property. Let $\alpha<\beta<\gamma$ be three pseudometric then we shall show that has no product property if it is given by a linear combination of α and γ.

PROPOSITION 2. *Let α and β be two pseudometrics given by Schwarz-Pick systems. Then the pseudometric $\gamma=t_1\alpha+t_2\beta$; $0<t_1, t_2<1$, $t_1+t_2=1$, has no product property.*

PROOF. Let S be a domain with $b_1, b_2\in S$, $\alpha(b_1, b_2)=M$, $\beta(b_1, b_2)=N$ and $M>N$. Let S be the open unit disk in the complex plane choose two points a_1 and a_2 in S with $\alpha(a_1, a_2)=K$, and $M>K>N$. Then we calculate the γ distance of $((a_1, b_1), (a_2, b_2))$ in $D\times S$.

$$\gamma((a_1, b_1), (a_2, b_2))=t_1\alpha((a_1, b_1), (a_2, b_2))+t_2\beta((a_1, b_1), (a_2, b_2))$$
$$\geqq \mathrm{Max}\{t_1K, t_1 M\}+t_2 \mathrm{Max}\{t_2K, t_2N\}=t_1K+t_2N.$$

But $\max\{\gamma(a_1a_2), (b_1, b_2)\}=\max\{K, t_1M+t_2N\}$.

Hence we know that $\max\gamma((a_1, b_1), (a_2, b_2))>\max\{\gamma(a_1, a_2), \gamma(b_1, b_2)\}$ hence we see that γ does not have product property, and we proved the statement.

NOTE. It is known that the Kobayashi pseudometric is strictly bigger than the Caratheodory pseudometric on any non-simply connected surfaces, see Kobayashi [3], or see [2]. But I have no example of a manifold on which all of the Kobayashi, the Caratheodory and the harmonic pseudometrics are different. It is known that the Kobayashi pseudometric has the product property, see [4]. It was impossible to tell whether the harmonic metric has the property or not, since we have no way to know that it is a proper linear combination of the other two pseudometrics.

Seoul National University

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