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# ON GENERIC SUBMANIFOLDS WITH ANTINORMAL STRUCTURE OF AN ODD-DIMENSIONAL SPHERE

*Dedicated to professor Chung-Ki Pahk on his sixty fifth birthday*

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## 0. Introduction

Recently, several authors have studied generic (anti-holomorphic) submanifold of a Kaehlerian manifold ([5], [7], [9], [10], [11], etc.)

On the other hand, the author in the previous paper ([4]) studied a generic submanifold of an odd-dimensional unit sphere under the condition that structure tensor $f$ induced on the submanifold is normal.

The purpose of the present paper is to study a minimal generic submanifold of an odd-dimensional unit sphere whose induced structure on the submanifold is antinormal (see **1**).

In **1**, we recall fundamental properties and structure equations for a generic submanifold immersed in a Sasakian manifold and define the structure tensor $f$ on the submanifold to be antinormal.

In **2**, we investigate a generic submanifold with antinormal structure of an odd-dimensional sphere whose normal connection is flat.

In the last **3**, we characterize minimal generic submanifolds of an odd-dimensional sphere under certain conditions.

## 1. Preliminaries

Let $M^{2m+1}$ be a $(2m+1)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{U, x^h\}$ and $(F_j^{\ h}, G_{ji}, F^h)$ the set of structure tensors of $M^{2m+1}$, where, here and in the sequel, the indices $h, j, i, \cdots$ run over the range $\{1', 2', \cdots, (2m+1)'\}$. Then we have

$$(1.1)\qquad F_t^{\ h}F_i^{\ t}=-\delta_i^h+F_iF^h,\ \ F_tF_i^{\ t}=0,\ \ F_t^{\ h}F^t=0,$$
$$F_tF^t=1,\ \ F_j^{\ t}F_i^{\ s}G_{ts}=G_{ji}-F_jF_i,$$

where $F_i=G_{it}F^t$ and

$$(1.2)\qquad \nabla_jF^h=F_j^{\ h},\ \ \nabla_jF_i^{\ h}=-G_{ji}F^h+\delta_j^{\ h}F_i,$$

$\nabla_j$ denoting the operator of covariant differentiation with respect to $G_{ji}$.

Let $M^n$ be an $n$-dimensional Riemannian manifold isometrically immersed in $M^{2m+1}$ by the immersion $i: M^n \to M^{2m+1}$ and identify $i(M^n)$ with $M^n$. In terms of local coordinates $(y^a)$ of $M^n$ and $(x^h)$ of $M^{2m+1}$ the immersion $i$ is locally expressed by $x^h = x^h(y^a)$, where, here and in the sequel, the indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, n\}$. If we put $B_b^{\ h} = \partial_b x^h$, $\partial_b = \partial/\partial y^b$, then $B_b^{\ h}$ are $n$ linearly independent vectors of $M^{2m+1}$ tangent to $M^n$. Denoting by $g_{cb}$ the fundamental metric tensor of $M^n$, we then have

$$g_{cb} = B_c^{\ h} B_b^{\ k} G_{hk}$$

because the immersion is isometric.

We denote by $C_x^{\ h}$ $2m+1-n$ mutually orthogonal unit normals of $M^n$ (the indices $u$, $v$, $w$, $x$, $y$ and $z$ run over the range $\{1^*, \cdots, (2m+1-n)^*\}$. Therefore, denoting by $\nabla_c$ the operator of van der Waerden-Bortolotti covariant differentiation with respect to the Christofel symbols $\left\{ {}_{c}{}^{a}{}_{b} \right\}$ formed with $g_{cb}$, we have equations of Gauss and Weingarten for $M^n$

$$\nabla_c B_b^{\ h} = h_{cb}^{\ \ x} C_x^{\ h}, \tag{1.3}$$

$$\nabla_c C_x^{\ h} = -h_c^{\ a}{}_x B_a^{\ h} \tag{1.4}$$

respectively, where $h_{cb}^{\ \ x}$ are the second fundamental tensors with respect to the normals $C_x^{\ h}$ and $h_c^{\ a}{}_x = h_{cb}^{\ \ y} g^{ab} g_{xy}$, $g_{xy}$ being the metric tensor of the normal bundle of $M^n$ given by $g_{xy} = G_{ji} C_x^{\ j} C_y^{\ i}$, and $(g^{cb}) = (g_{cb})^{-1}$.

If the ambient manifold $M^{2m+1}$ is a $(2m+1)$-dimensional unit sphere $S^{2m+1}(1)$, the equations of Gauss, Codazzi and Ricci for $M^n$ are respectively given by

$$K_{dcb}^{\ \ \ a} = \delta_d^a g_{cb} - \delta_c^{\ a} g_{db} + h_d^{\ ax} h_{cbx} - h_c^{\ ax} h_{dbx}, \tag{1.5}$$

$$\nabla_d h_{cb}^{\ \ x} - \nabla_c h_{db}^{\ \ x} = 0, \tag{1.6}$$

$$K_{dcy}^{\ \ \ x} = h_{de}^{\ \ x} h_c^{\ e}{}_y - h_{ce}^{\ \ x} h_d^{\ e}{}_y, \tag{1.7}$$

where $K_{dcb}^{\ \ \ a}$ and $K_{dcy}^{\ \ \ x}$ are the Riemannian Christoffel curvature tensor of $M^n$ and that of the connection induced in the normal bundle respectively.

Now we consider the submanifold $M^n$ of $M^{2m+1}$ which satisfy

$$N_P(M^n) \perp F(N_P(M^n))$$

at each point $P\in M^n$, where $N_P(M^n)$ denotes the normal space at $P$. Such a submanifold is called a *generic submanifold* (an *anti-holomorphic submanifold*), ([4], [8], [10]).

From now on we consider in the sequel generic submanifolds immersed in a Sasakian manifold $M^{2m+1}$. Then we can put in each coordinate neighborhood

$$F_t{}^h B_c{}^t = f_c{}^a B_a{}^h - f_c{}^x C_x{}^h, \tag{1.8}$$

$$F_t{}^h C_x{}^t = f_x{}^a B_a{}^h, \tag{1.9}$$

$$F^h = v^a B_a{}^h + u^x C_x{}^h, \tag{1.10}$$

where $f_c{}^a$ is a tensor field of type (1, 1) defined on $M^n$, $f_c{}^x$ a local 1-form for each fixed index $x$, $v^a$ a vector field, $u^x$ a function for each fixed index $x$, and $f_x{}^a = f_c{}^y g^{ac} g_{yx}$.

Applying $F$ to (1.8) and (1.9) respectively, and using (1.1) and these equations, we easily find that ([4])

$$\begin{cases} f_c{}^e f_e{}^a = -\delta_c^a + f_c{}^x f_x{}^a + v_c v^a, \\ f_c{}^e f_e{}^x = -v_c u^x, \\ f_x{}^e f_e{}^y = \delta_x^y - u_x u^y, \\ v^e f_e{}^a = -u^x f_x{}^a, \\ v^c f_e{}^x = 0, \\ g_{de} f_c^d f_b^e = g_{cb} - f_c{}^x f_{xb} - v_c v_b. \end{cases} \tag{1.11}$$

Since $F_t F^t = 1$, we have from (1.10)

$$v_a v^a + u_x u^x = 1. \tag{1.12}$$

Putting $f_{cb} = f_c{}^a g_{ab}$, $f_{cx} = f_c{}^y g_{yx}$, then we easily see that $f_{cb} = -f_{bc}$, $f_{cx} = f_{xc}$ because of (1.11).

Differentiating (1.8), (1.9) and (1.10) covariantly and using (1.1)~(1.4), we have respectively (see [4])

$$\nabla_c f_b{}^a = -g_{cb} v^a + \delta_c{}^a v_b + h_{cb}{}^x f_x{}^a - h_c{}^a{}_x f_b{}^x, \tag{1.13}$$

$$\nabla_c f_b{}^x = g_{cb} u^x + h_{ce}{}^x f_b{}^e, \tag{1.14}$$

$$\nabla_c f_x{}^a = \delta_c^a u_x + h_{ecx} f^{ae}, \tag{1.15}$$

(1.16) $$h_c{}^e{}_x f_e{}^y = h_c{}^{ey} f_{ex},$$

(1.17) $$\nabla_c v_b = f_{cb} + h_{cb}{}^x u_x,$$

(1.18) $$\nabla_c u^x = -f_c{}^x - h_{ce}{}^x v^e$$

with the aid of (1.8)~(1.10), where $h_{cbx} = h_{cb}{}^y g_{yx}$ and $f^{ae} = f_c{}^e g^{ca}$.

When $M^n$ is a hypersurface of $M^{2m+1}$, (1.11) and (1.12) reduce to

$$\begin{cases} f_c{}^e f_e{}^a = -\delta_c{}^a + u_c u^a + v_c v^a, \\ f_c{}^e u_e = -\lambda v_c, \quad v^e f_e{}^a = -\lambda u^a, \\ v^e u_e = 0, \quad u_e u^e = 1-\lambda^2, \\ g_{de} f_c{}^d f_b{}^e = g_{cb} - u_c u_b - v_c v_b, \\ v_a v^a = 1-\lambda^2, \end{cases}$$

where we have put $f_c^{1^*} = u_c$, $u_{1^*} = u^{1^*} = \lambda$. These mean that the set $(f_b{}^a, g_{cb}, u_b, v_b, \lambda)$ defines the so-called $(f, g, u, v, \lambda)$-structure on $M^{2m}$ ([2], [6]).

The aggregate $(f_c{}^a, g_{cb}, f_c{}^x, v_c, u^x)$ satisfying (1.11) and (1.12) is said to be *antinormal* if

(1.19) $$h_c{}^{ex} f_e{}^a + f_c{}^e h_e{}^{ax} = 0.$$

In characterizing the submanifold, we shall use the following Theorem A ([1], [2]).

THEOREM A. *Let $M^{2m}$ be a complete hypersurface with antinormal $(f, g, u, v, \lambda)$-structure of an odd-dimensional unit sphere $S^{2m+1}(1)$. If the function $\lambda$ does not vanish almost everywhere and the scalar curvature of $M^{2m}$ is a constant, then $M^{2m}$ is a great sphere $S^{2m}(1)$ or a product of two spheres $S^m(1/\sqrt{2}) \times S^m(1/\sqrt{2})$.*

## 2. Generic submanifolds with antinormal structure

Throughout this paper we assume that the induced structure satisfying (1.11) and (1.12) is antinormal. We then have from (1.19)

(2.1) $$h_{ce}{}^x f_b{}^e = h_{be}{}^x f_c{}^e.$$

Transvecting (2.1) with $f_a{}^b$ and the first relation of (1.11), we find

$$h_{ce}{}^{x}(-\delta_{a}^{e}+f_{a}{}^{z}f_{z}{}^{e}+v_{a}v^{e})=h_{be}{}^{x}f_{c}{}^{e}f_{a}{}^{b},$$

from which, taking the skew-symmetric part,

$$(h_{ce}{}^{x}f_{z}{}^{e})f_{b}{}^{z}-(h_{be}{}^{x}f_{z}{}^{e})f_{c}{}^{z}+(h_{ce}{}^{x}v^{e})v_{b}-(h_{be}{}^{x}v^{e})v_{c}=0. \tag{2.2}$$

If we transvect (2.2) with $v^{b}$ and take account of (1.11) and (1.12), then we obtain

$$-(h_{be}{}^{x}v^{b}f_{z}{}^{e})f_{c}{}^{z}+(1-A^{2})h_{ce}{}^{x}v^{e}-(h_{de}{}^{x}v^{d}v^{e})v_{c}=0, \tag{2.3}$$

where $A^{2}=u_{x}u^{x}$, which transvect $f_{y}{}^{c}$ and use (1.11),

$$A^{2}h_{ce}{}^{x}v^{e}f_{y}{}^{c}=(h_{be}{}^{x}v^{b}f_{z}{}^{e}u^{z})u_{y}.$$

Thus, we have from (2.3)

$$A^{2}(1-A^{2})h_{ce}{}^{x}v^{e}=A^{2}(h_{de}{}^{x}v^{d}v^{e})v_{c}+(h_{be}{}^{x}v^{b}f_{z}{}^{e}u^{z})u_{y}f_{c}{}^{y}. \tag{2.4}$$

Now we suppose in the sequel that the function $A$ does not vanish almost everywhere and $n\neq m$, then so does $A(1-A^{2})$. In fact, if $1-A^{2}$ vanishes identically, then we see from (1.12) that $v_{c}=0$ and hence $f_{cb}=0$ because of (1.17). Thus we verify that

$$0=f_{cb}f^{cb}=2(n-m)$$

with the aid of (1.11) and (1.12). Therefore $A(1-A^{2})$ is nonzero almost everywhere.

Consequently (2.4) implies

$$h_{ce}{}^{x}v^{e}=B^{x}v_{c}+A^{x}u_{z}f_{c}{}^{z}, \tag{2.5}$$

where we have put

$$A^{x}=(h_{de}{}^{x}v^{d}f_{z}{}^{e}u^{z})/A^{2}(1-A^{2}),\quad B^{x}=(h_{de}{}^{x}v^{d}v^{e})/(1-A^{2}).$$

Substituting (2.5) into (2.2), we find

$$(h_{ce}{}^{x}f_{z}{}^{e})f_{b}{}^{z}-(h_{be}{}^{x}f_{z}{}^{e})f_{c}{}^{z}+A^{x}(u_{z}f_{c}{}^{z}v_{b}-u_{z}f_{b}{}^{z}v_{c})=0,$$

from which, transvecting $f_{y}{}^{b}$ and making use of (1.11),

$$h_{ce}{}^{x}f_{y}{}^{e}-(h_{ce}{}^{x}f_{z}{}^{e}u^{z})u_{y}-(h_{de}{}^{x}f_{z}{}^{e}f_{y}{}^{d})f_{c}{}^{z}-(1-A^{2})A^{x}u_{y}v_{c}=0. \tag{2.6}$$

On the other hand, we have

$$\begin{aligned} h_{ce}{}^{x}f_{z}{}^{e}u^{z}=-h_{ce}{}^{x}v^{a}f_{a}{}^{e}=-h_{ae}{}^{x}v^{a}f_{c}{}^{e}&=-(B^{x}v_{e}+A^{x}u_{y}f_{e}{}^{y})f_{c}{}^{e}\\ &=-B^{x}u_{z}f_{c}{}^{z}+A^{2}A^{x}v_{c}\end{aligned}$$

with the help of (1.11), (2.1) and (2.5). Thus (2.6) reduces to

$$h_{ce}{}^{x} f_{y}{}^{e} = Q_{yz}{}^{x} f_{c}{}^{z} + A^{x} u_{y} v_{c}, \tag{2.7}$$

where we have put

$$Q_{yz}{}^{x} = h_{de}{}^{x} f_{z}{}^{e} f_{y}{}^{d} - B^{x} u_{z} u_{y}, \tag{2.8}$$

which implies

$$Q_{yz}{}^{x} = Q_{zy}{}^{x}.$$

Putting $Q_{yzx} = Q_{yz}{}^{w} g_{wx}$, we see from (2.7) that

$$(Q_{yzx} - Q_{xzy}) f_{c}{}^{z} + (A_{x} u_{y} - A_{y} u_{x}) v_{c} = 0 \tag{2.9}$$

because of (1.16).

Transvection $v^{c}$ and $f_{a}{}^{c}$ give respectively

$$A_{x} u_{y} - A_{y} u_{x} = 0, \tag{2.10}$$

$$(Q_{yzx} - Q_{xzy}) u^{z} = 0 \tag{2.11}$$

because $1 - A^{2}$ does not vanish almost everywhere.

Transvecting also (2.9) with $f_{w}{}^{c}$ and using (1.11) and (2.11), we obtain $Q_{yzx} = Q_{xzy}$. Hence $Q_{xyz}$ is symmertric for any index.

Transvecting (2.7) with $f_{a}{}^{c}$ and taking account of (1.11), we find

$$h_{ce}{}^{x} f_{y}{}^{e} f_{a}{}^{c} = -Q_{yz}{}^{x} u^{z} v_{a} + A^{x} u_{y} (u_{z} f_{a}{}^{z}),$$

from which, using (1.11), (2.1) and (2.5),

$$Q_{yz}{}^{x} u^{z} + B^{x} u_{y} = 0. \tag{2.12}$$

This implies

$$B_{x} u_{y} - B_{y} u_{x} = 0 \tag{2.13}$$

because $Q_{xyz}$ is symmetric for all indices.

$A$ being nonzero almost everywhere, (2.10) and (2.13) give respectively

$$A^{x} = \beta u^{x}, \quad B^{x} = \alpha u^{x}, \tag{2.14}$$

where

$$\beta = A^{x} u_{x} / A^{2}, \quad \alpha = B^{x} u_{x} / A^{2}. \tag{2.15}$$

Thus (2.5), (2.7) and (2.12) reduce respectively to

$$h_{ce}{}^{x} v^{e} = u^{x} (\alpha v_{c} + \beta u_{z} f_{c}{}^{z}), \tag{2.16}$$

(2.17) $$h_{ce}{}^{x}f_{y}{}^{e}=Q_{yz}{}^{x}f_{c}{}^{z}+\beta u^{x}u_{y}v_{c},$$

(2.18) $$Q_{yz}{}^{x}u^{z}=-\alpha u^{x}u_{y}.$$

Transvecting (2.1) with $f^{cb}$ yields

$$\begin{aligned}0&=h_{ce}{}^{x}(-g^{ce}+f^{cz}f_{z}{}^{e}+v^{c}v^{e})\\&=-h^{x}+Q_{yz}{}^{x}(g^{zy}-u^{z}u^{y})+\alpha u^{x}(1-A^{2})\\&=-h^{x}+Q^{x}+\alpha u^{x}\end{aligned}$$

with the aid of (1.11) and (2.16)~(2.18), where $h^{x}=g^{cb}h_{cb}{}^{x}$ and $Q^{x}=g^{yz}Q_{yz}{}^{x}$. Hence, it follows that

(2.19) $$h^{x}=Q^{x}+\alpha u^{x}.$$

We now assume that the connection induced in the normal bundle of $M^{n}$ is flat, that is, $K_{dcy}{}^{x}=0$. Then we have from (1.7)

(2.20) $$h_{ce}{}^{x}h_{b}{}^{e}{}_{y}=h_{be}{}^{x}h_{c}{}^{e}{}_{y}.$$

Transvecting (2.20) with $f_{z}{}^{b}$ and using (2.17), we get

$$h_{ce}{}^{x}(Q_{wyz}f^{ew}+\beta u_{y}u_{z}v^{e})=h_{c}{}^{e}{}_{y}(Q_{wz}{}^{x}f_{e}{}^{w}+\beta u^{x}u_{z}v_{e}),$$

from which, using (2.16) and (2.17),

$$Q_{wyz}(Q_{v}{}^{wx}f_{c}{}^{v}+\beta u^{x}u^{w}v_{c})+\alpha\beta u^{x}u_{y}u_{z}v_{c}=Q_{wz}{}^{x}(Q_{vy}{}^{w}f_{c}{}^{v}+\beta u^{w}u_{y}v_{c})+\alpha\beta u^{x}u_{y}u_{z}v_{c},$$

or, using (2.18),

(2.21) $$Q_{z}\;Q_{v}{}^{wx}f_{c}{}^{v}=Q_{wz}{}^{x}Q_{vy}{}^{w}f_{c}{}^{v}.$$

Transvecting (2.21) with $f_{a}{}^{c}$ and $f_{u}{}^{c}$ and taking account of (1.11), we have respectively

$$Q_{wyz}Q_{v}{}^{wx}(u^{v}v_{a})=Q_{wz}{}^{x}Q_{vy}{}^{w}(u^{v}v_{a}),$$

$$Q_{wyz}Q_{v}{}^{wx}(\delta_{u}{}^{v}-u_{u}u^{v})=Q_{wz}{}^{x}Q_{vy}{}^{w}(\delta_{v}{}^{u}-u_{u}u^{v}).$$

The last two relationships give

(2.22) $$Q_{wyz}Q_{vx}{}^{w}=Q_{wzx}Q_{vy}{}^{w}$$

because $1-A^{2}$ does not vanish almost everywhere, which implies

(2.23) $$Q_{yzx}Q^{yzx}=Q_{x}Q^{x}.$$

LEMMA 2.1. *Let $M^{n}$ be a generic submanifold of $S^{2m+1}(1)$, $(n\neq m,\ n\neq m+1)$*

*whose normal connection is flat. If the induced structure on $M^n$ is antinormal and the function $u_x u^x$ is nonzero almost everywhere. Then we have*

$$h_{ce}{}^{x}v^{e}=\beta u^{x}u_{z}f_{c}{}^{z}, \tag{2.24}$$

$$Q_{yz}{}^{x}u^{z}=0,\quad h^{x}=Q^{x}. \tag{2.25}$$

PROOF. From (2.16) we have

$$h_{ce}{}^{x}v^{e}u_{x}=\alpha A^{2}v_{c}+\beta A^{2}(u_{x}f_{c}{}^{x}).$$

Differentiating this covariantly and substituting (1.14), (1.17) and (1.18), we obtain

$$(\nabla_{d}h_{ce}{}^{x})v^{e}u_{x}+h_{c}{}^{ex}u_{x}(f_{de}+h_{de}{}^{y}u_{y})-h_{ce}{}^{x}v^{e}(f_{dx}+h_{dax}v^{a})=\nabla_{d}(\alpha A^{2})v_{c}+\nabla_{d}(\beta A^{2})u_{x}f_{c}{}^{x}$$
$$+\alpha A^{2}(f_{dc}+h_{dc}{}^{x}u_{x})-\beta A^{2}f_{c}{}^{x}(f_{dx}+h_{dex}v^{e})+\beta A^{2}u_{x}(g_{dc}u^{x}+h_{de}{}^{x}f_{c}{}^{e}),$$

from which, taking the skew-symmetric part and using (2.16) and (2.20),

$$\nabla_{d}(\alpha A^{2})v_{c}-\nabla_{c}(\alpha A^{2})v_{d}+\nabla_{d}(\beta A^{2})u_{x}f_{c}{}^{x}-\nabla_{c}(\beta A^{2})u_{x}f_{d}{}^{x}+2\alpha A^{2}f_{dc}$$
$$+\alpha(\beta A^{2}+1)(u_{x}f_{d}{}^{x}v_{c}-u_{x}f_{c}{}^{x}v_{d})=0 \tag{2.26}$$

with the aid of (1.6) and (2.1). If we transvect (2.26) with $v^{c}$ and take account of (1.11) and (1.12), we get

$$(1-A^{2})\nabla_{d}(\alpha A^{2})=v^{e}\nabla_{e}(\alpha A^{2})v_{d}+\{v^{e}\nabla_{e}(\beta A^{2})$$
$$-2\alpha A^{2}-\alpha(\beta A^{2}+1)(1-A^{2})\}u_{x}f_{d}{}^{x}. \tag{2.27}$$

Transvecting also (2.26) with $f_{z}{}^{c}$ and using (1.11), we find

$$(1-A^{2})\nabla_{d}(\beta A^{2})u_{z}=f_{z}{}^{e}\nabla_{e}(\alpha A^{2})v_{d}+f_{z}{}^{e}\nabla_{e}(\beta A^{2})u_{x}f_{d}{}^{x}+2\alpha A^{2}u_{z}v_{d}$$
$$+\alpha(\beta A^{2}+1)(1-A^{2})u_{z}v_{d}.$$

Hence, the last two equations give

$$A^{2}(1-A^{2})\nabla_{d}(\beta A^{2})=A^{2}v^{e}\nabla_{e}(\beta A^{2})v_{d}+u^{z}f_{z}{}^{e}\nabla_{e}(\beta A^{2})u_{x}f_{d}{}^{x}. \tag{2.28}$$

Substituting (2.27) and (2.28) into (2.26), we get

$$\alpha\{(1-A^{2})f_{dc}-(u_{x}f_{d}{}^{x}v_{c}-u_{x}f_{c}{}^{x}v_{d})\}=0$$

because $A(1-A^{2})$ does not vanish almost everywhere, from which, transvecting $f^{dc}$ and making use of (1.11) and (1.12),

$$\alpha\{n-(2m+1-n)+A^{2}-1+A^{2}\}+2\alpha u_{x}u_{y}(g^{xy}-u^{x}u^{y})=0,$$

that is, $\alpha(n-m-1)=0$. Since $n\neq m+1$, we have $\alpha=0$. Therefore, (2.16), (2.18) and (2.19) reduce to (2.24) and (2.25).

## 3. Minimal generic submanifolds with antinormal structure

In this section we consider a minimal generic submanifold $M^n$ of an odd-dimensional unit sphere $S^{2m+1}(1)$.

First of all we prove

LEMMA 3.1. *Let $M^n$ be a minimal generic submanifold of $S^{2m+1}(1)$, ($n\neq m$, $n\neq m+1$) whose normal connection is flat. If the induced structure on $M^n$ is antinormal and the function A is nonzero almost everywhere. Then we have*

$$(3.1)\qquad (1-A^2)h_{cex}h_b{}^e{}_y=u_xu_y\{\beta(1-A^2)(g_{cb}-f_c{}^zf_{zb})$$
$$+\beta(\beta-1)(u_zf_b{}^z)(u_wf_c{}^w)+\beta(\beta A^2-1)v_cv_b\}.$$

PROOF. Since $M^n$ is minimal, we see from the second equation of (2.25) that $Q^x=0$. Thus (2.17) becomes

$$(3.2)\qquad h_{cex}f_y{}^e=\beta u_xu_yv_c$$

with the aid of (2.23). Differentiating (3.2) covariantly and substituting (1.14), (1.17) and (1.18), we find

$$(\nabla_dh_{cex})f_y{}^e+h_c{}^e{}_x(g_{de}u_y+h_{day}f_e{}^a)=(\nabla_d\beta)u_xu_yv_c-\beta u_yv_c(f_{dx}+\beta u_xu_zf_d{}^z)$$
$$-\beta u_xv_c(f_{dy}+\beta u_yu_zf_d{}^z)+\beta u_xu_y(f_{dc}+h_{dcz}u^z)$$

because of (2.24), from which, taking the skew-symmetric part with respect to $d$ and $c$ and using (1.6),

$$(3.3)\qquad 2h_{ce}{}^xh_{day}f_e{}^a=\{(\nabla_d\beta)v_c-(\nabla_c\beta)v_d\}u_xu_y+\beta\{v_d(u_xf_{cy}+u_yf_{cx})$$
$$-v_c(u_xf_{dy}+u_yf_{dx})\}+2\beta f_{dc}u_xu_y-2\beta^2(v_cu_zf_d{}^z-v_du_zf_c{}^z)u_xu_y.$$

If we transvect (3.3) with $v^c$ and take account of (1.11), (2.24) and (3.2), we obtain

$$(3.4)\qquad (1-A^2)(\nabla_d\beta)u_xu_y=(v^e\nabla_e\beta)v_du_xu_y+\beta(1-A^2)(u_xf_{dy}+u_yf_{dx})$$
$$+2\beta(\beta-1)(u_zf_d{}^z)u_xu_y.$$

Substitution the above equation into (3.3) yields

$$(3.5)\quad (1-A^2)h_c{}^e{}_xh_{day}f_e{}^a=u_xu_y\{\beta(\beta A^2-1)(vcu_zf_d{}^z-v_du_zf_c{}^z)+\beta(1-A^2)f_{dc}u_xu_y\}.$$

Transvecting (3.5) with $f_b^{\ d}$ and making use of (1.11) and (2.1), we get

$$(1-A^2)h_c{}^e{}_x h_{aey}(-\delta_b^{\ a}+f_b^{\ z}f_z^{\ a}+v_b v^a)=-\beta(\beta A^2-1)\{A^2 v_c v_b+(u_z f_c^{\ z})(u_w f_b^{\ w})\}u_x u_y$$
$$+\beta(1-A^2)(-g_{cb}+f_c^{\ z}f_{zb}+v_c v_b)u_x u_y,$$

or, using (2.24) and (3.2),

$$(1-A^2)\{-h_c{}^e{}_x h_{bey}+\beta^2 u_x u_y(u_z f_c^{\ z})(u_w f_b^{\ w})+\beta^2 A^2 u_x u_y v_c v_b\}$$
$$=-\beta(\beta A^2-1)\{A^2 v_c v_b+(u_z f_c^{\ z})(u_w f_b^{\ w})\}u_x u_y+\beta(1-A^2)(-g_{cb}+f_c^{\ z}f_{zb}+v_c v_b)u_x u_y.$$

Thereby, (3.1) Perived from this. This completes the proof of the lemma.

From (3.1) we have

$$(3.6)\qquad (1-A^2)h_{cex}h_b^{\ ex}=\beta A^2\{(1-A^2)(g_{cb}-f_c^{\ z}f_{zb})$$
$$+(\beta-1)(u_x f_c^{\ x})(u_y f_b^{\ y})+(\beta A^2-1)v_c v_b\}.$$

Since we have

$$g^{cb}(g_{cb}-f_c^{\ z}f_{bz})=2n-2m-1+A^2,\quad (u_x f_c^{\ x})(u_y f^{cy})=A^2(1-A^2),$$

(3.6) implies

$$(3.7)\qquad h_{cb}^{\ \ x}h^{cb}{}_x=2\beta A^2(n-m-1+\beta A^2).$$

LEMMA 3.2. *Under the same assumptions as those stated in Lemma* 3.1, *if the scalar curvature of $M^n$ is a constant, then we have*

$$(3.8)\qquad \beta A^2(\beta A^2-1)=0.$$

PROOF. We see from (1.5) that

$$(3.9)\qquad K_{cb}=(n-1)g_{cb}+h_c^{\ ex}h_{bex}$$

because $M^n$ is minimal, where $K_{cb}$ is the Ricci tensor of $M^n$. Thus, the scalar curvature $K$ of $M^n$ is given by

$$K=n(n-1)-h_{cb}^{\ \ x}h^{cb}{}_x.$$

From this fact and (3.7) we see that

$$K=n(n-1)-2\gamma(n-m-1+\gamma)$$

where $\gamma=\beta A^2$. Since $K$ is a constant, by differentiating covariantly, we find

$$(n-m+2\gamma)\nabla_c\gamma=0,$$

which means that $\gamma$ is a constant on $M^n$.

On the other hand, we have from (3.2)

$$h_{ce}{}^{x} f_{x}{}^{e} = \gamma v_{c}.$$

Differentiating this covariantly and using (1.14) and (1.17), we get

$$(\nabla_{d} h_{ce}{}^{x}) f_{x}{}^{e} + h_{c}{}^{ex}(g_{de} u_{x} + h_{dax} f_{e}{}^{a}) = \gamma(f_{dc} + h_{dc}{}^{x} u_{x})$$

because $\gamma$ is a constant, from which, taking the skew-symmetric part and using (1.6),

$$h_{ce}{}^{x} h_{dax} f_{a}{}^{e} = \gamma f_{dc}.$$

Tranvection $v^{c}$ yields $\gamma^{2} u_{x} f_{d}{}^{x} = \gamma u_{x} f_{d}{}^{x}$ with the aid of (2.24) and (3.2), which gives $\gamma(\gamma-1)=0$ because $A(1-A^{2})$ does not vanish almost everywhere. Hence the lemma is proved.

Finally we prove

THEOREM 3.3. *Let $M^{n}$ be a complete and minimal generic submanifold of an odd-dimensional unit sphere $S^{2m+1}(1)$, ($n \fallingdotseq m$, $n \fallingdotseq m+1$) whose normal connection is flat. If the induced structure on $M^{n}$ is antinormal and the scalar curvature of $M^{n}$ is a constant, then $M^{n}$ is a great spere $S^{n}$ or a product of two spheres $S^{m}(1/\sqrt{2}) \times S^{m}(1/\sqrt{2})$ provided that the function $u^{x} u_{x}$ does not vanish almost everywhere.*

PROOF. If the case in which $\beta A^{2}=0$, then we see from (3.7) that $h_{cb}{}^{x}=0$. By completeness of $M^{n}$, $M^{n}$ is a great sphere $S^{n}$.

From Lemma 3.2 it remains the case in which $\beta A^{2}=1$. In this case, (3.1) reduces to

$$h_{cex} h_{b}{}^{e}{}_{y} = (\beta-1)/(1-A^{2}) u_{x} u_{y} \{g_{cb} - f_{c}{}^{z} f_{zb} + \beta(u_{z} f_{c}{}^{z})(u_{w} f_{b}{}^{w})\}, \tag{3.10}$$

which implies

$$h_{cb}{}^{x} h^{cb}{}_{x} = 2(n-m). \tag{3.11}$$

On the other hand, from the Ricci identity

$$\nabla_{d} \nabla_{c} h_{bz}{}^{x} - \nabla_{c} \nabla_{d} h_{ba}{}^{x} = -K_{dcb}{}^{e} h_{ae}{}^{x} - K_{dca}{}^{e} h_{be}{}^{x}$$

we have

$$(g^{da} \nabla_{d} \nabla_{a} h_{cb}{}^{x}) h^{cb}{}_{x} = K_{ce} h_{b}{}^{ey} h^{cb}{}_{y} - K_{dcba} h^{day} h^{cb}{}_{y}, \tag{3.12}$$

where $K_{dcba} = K_{dcb}{}^{e} g_{ea}$.

From (1.5) we find

$$K_{dcba}h^{day}h^{cb}{}_y = -h_{cbx}h^{cbx} + (h_{dax}h^{da}{}_y)(h_{cb}{}^x h^{cby}) - (h_{cax}h^{cb}{}_y)(h_{db}{}^x h^{day}),$$

from which, using (2.24), (3.2), (3.8) and (3.10),

$$K_{dcba}h^{day}h^{cb}{}_y = 4(n-m)(n-m-1). \tag{3.13}$$

We have from (3.9) and (3.10)

$$K_{cb} = (n-2)g_{cb} + f_c{}^z f_{bz} - \beta(u_x f_c{}^x)(u_y f_b{}^y).$$

Transvection $h_a{}^{by}h^{ca}{}_y$ gives

$$K_{ce}h_b{}^{ey}h^{cb}{}_y = 2(n-2)(n-m) \tag{3.14}$$

with the aid of (2.24), (3.2), (3.10) and the fact that $\beta A^2=1$.

Substituting (3.12) into the identity

$$\frac{1}{2}\Delta(h_{cb}{}^x h^{cb}{}_x) = g^{da}(\nabla_d\nabla_a h_{cb}{}^x)h^{cb}{}_x + \|\nabla_d h_{cb}{}^x\|^2 \tag{3.15}$$

and taking account of (3.11), (3.13) and (3.14), we obtain

$$2(n-m)(2m-n) + \|\nabla_d h_{cb}{}^x\|^2 = 0,$$

where $\Delta = g^{cb}\nabla_c\nabla_b$. This implies that $2m=n$ and $\nabla_d h_{cb}{}^x = 0$ because of $n \neq m$. The first assertion means that $M^n$ is a hypersurfaces of $S^{2m+1}(1)$. Thus, according to Therorem A, $M^n$ is $S^m(1/\sqrt{2}) \times S^m(1/\sqrt{2})$. Therefore the theorem is proved.

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