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ON $LC-$, $RC-$, AND C-LOOPS

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In this paper, identities which arise from $LC-$, $RC-$, and C-loop identities are considered. These give us characterizations of groups. In the first section some preliminary information is given. The second section consists $LC-$ and $RC-$ loop type identities while the last section contains C-loop type identity.

1. Preliminary information.

If $(G, \cdot)$ is a loop with identity 1, then for each $x \in G$ the mappings $R(x)$ and $L(x)$ are defined by $yR(x)=yx$ and $yL(x)=xy$ for all $y \in G$. It follows that $R(x)$ and $L(x)$ are permutations of G. If $(G, \cdot)$ is an inverse property loop, then corresponding to each $x \in G$, there is an $xJ=x^{-1}$ in G such that $xx^{-1}=x^{-1}x=1$, $R(x)^{-1}=R(x^{-1})$ and $L(x)^{-1}=L(x^{-1})$. Further $J^2=I$, $JL(x)J=R(x)^{-1}$, $JR(x)J=L(x)^{-1}$. J is known as inverse mapping. The left nucleus N_λ, the middle nucleus N_μ and the right nucleus N_ρ of a loop $(G, \cdot)$ are defined by

$$N_\lambda = \{x \in G \mid x \cdot yz = xy \cdot z, \text{ all } y, z \in G\},$$

$$N_\mu = \{y \in G \mid x \cdot yz = xy \cdot z, \text{ all } x, z \in G\},$$

$$N_\rho = \{z \in G \mid x \cdot yz = xy \cdot z \text{ all } x, y \in G\}.$$

The nucleus N of $(G, \cdot)$ is $N=N_\lambda \cap N_\mu \cap N_\rho$. An ordered triple (U, V, W) of one-to-one mappings U, V, and W of G onto G is called an autotopism of G if and only if $xU \cdot yV=(xy)W$ for all $x, y \in G$.

For detailed account of above mentioned concepts see Bruck [1]. We will require the following results.

THEOREM 1.1. *If* $(G, \cdot)$ *is an inverse property loop. Then* (i) *right, middle and left nuclei of* $(G, \cdot)$ *coincide with the nucleus of* $(G, \cdot)$ (*See Bruck* [1], P. 114).

(ii) *If* (U, V, W) *is an autotopism of* $(G, \cdot)$, *then* (JUJ, W, V) *and* (W, JVJ, U) *are also the autotopisms* $(G, \cdot)$ (*See Bruck* [1], P. 112).

2. $LC-$and RC-type

Let $(G, \cdot)$ be a loop. Then it is known that in $(G, \cdot)$ following identities are equivalent:

(i) $(x\cdot xy)z=x(x\cdot yz)$,

(ii) $(xx\cdot y)z=x(x\cdot yz)$,

(iii) $x\ x\cdot yz=(x\cdot xy)z$.

A loop $(G,\cdot)$ satisfying any one (and hence all) of these identities is called *LC-loop* [2]. The equivalence of the following identities is also known:

(iv) $(yz\cdot x)x=y(zx\cdot x)$,

(v) $(yz\cdot x)x=y(z\cdot xx)$,

(vi) $yz\cdot xx=y(zx\cdot x)$.

A loop $(G,\cdot)$ satisfying any one (and hence all) of these identities is called *RC-loop* [2]. Corresponding to the identities (i)→(iii), we consider the following identical relations:

(1) $(x\cdot(\alpha x\cdot y))z=x(\alpha x\cdot yz)$,

(2) $((x\cdot\alpha x)\cdot y)z=x(\alpha x\cdot yz)$,

(3) $(x\cdot\alpha x)\cdot yz=(x\cdot(\alpha x\cdot y))z$

where α is any mapping of G onto G. And corresponding to the identities (iv) →(vi), we consider the following identical relations:

(4) $(yz\cdot x)\cdot\beta x=y(zx\cdot\beta x)$,

(5) $(yz\cdot x)\cdot\beta x=y(z(x\cdot\beta x))$,

(6) $yz\cdot(x\cdot\beta x)=y(zx.\beta x)$

where β is any mapping of G onto G.

LEMMA 2.1. *For a loop $(G,\cdot)$ the identities (1), (2) and (3) are equivalent.*

PROOF. Assume (1) holds. Then $y=1$ in (1) gives $(x\cdot\alpha x)z=x(\alpha x\cdot z)$ for all $z\in G$. Therefore, from (1) we have (2). Similarly $z=1$ in (2) gives (3). Assume (3) holds. Then $z=1$ in (3) gives $(x\cdot\alpha x)y=x(\alpha x\cdot y)$ for all $y\in G$. Therefore, $x(\alpha x\cdot yz)=(x\cdot\alpha x)\cdot yz=(x\cdot(\alpha x\cdot y))z$ which is (1). This completes the lemma.

LEMMA 2.2. *A loop $(G,\cdot)$ satisfies (1) (and hence (2) and (3)) if and only if $(L(\alpha x)L(x),\ I, L(\alpha x)L(x))$ is an autotopism.*

PROOF. Follows directly from (1).

THEOREM 2.3. *If (1) holds in a loop $(G,\cdot)$, then*

(a) *$(G,\cdot)$ has the left inverse property,*

(b) *δx is in the left nucleus of $(G,\cdot)$ for all x in G, where $\delta: G\to G$* is a mapping such that $\delta x=x\cdot\alpha x$.

PROOF. (a) Let ${}^{-1}x,\ x^{-1}\in G$ such that

(7) $$^{-1}x\cdot x=1=x\cdot x^{-1}, \text{ for all } x\in G.$$

Putting $y=(\alpha x)^{-1}$ in (1), we have $z=\alpha x\ ((\alpha x)^{-1}\cdot z)$
that is,

(8) $$z=u(u^{-1}\cdot z), \text{ for all } u, z\in G.$$

Setting $z=u$ in (8), we see that $u^{-1}={}^{-1}u$ and $(u^{-1})^{-1}=u$. So, (8) gives left inverse property.

(b) From (1), $y=1$ gives $(x\cdot\alpha x)z=x(\alpha x\cdot z)$ i.e. $L(x\cdot\alpha x)=L(\alpha x)L(x)$. Therefore, Lemma 2.2 gives that $(L(\delta x), I, L(\delta x))$ is an autotopism, showing thereby that $\delta x\in N_\lambda$(left nucleus). This completes the proof of this Theorem.

COROLLARY 2.4. *If* (1) *holds in a loop* $(G,\cdot)$ *with* $\delta x=x^2$, *for all* $x\in G$, *then* $(G,\cdot)$ *is LC-loop.*

THEOREM 2.5. *If* $(G,\cdot)$ *is LC-loop with* $\theta x\in N$, *for all* $x\in G$, *where* $\theta: G\to G$ *is a mapping such that* $\alpha x=x\cdot\theta x$. Then $(G,\cdot)$ *satisfies* (3) *and hence* (1) *and* (2).

PROOF. From (iii) and the fact that $\theta x\in N$. we get

$$\begin{aligned}(x\cdot\alpha x)\cdot yz&=[x\cdot(x\cdot\theta x)]\cdot yz=xx\cdot(\theta x\cdot yz)\\&=xx\cdot[(\theta x\cdot y)\cdot z]=[x\cdot(x\cdot(\theta x\cdot y))]z\\&=[x\cdot((x\cdot\theta x)\cdot y)]z=(x\cdot(\alpha x\cdot y))z\end{aligned}$$

which is (3) and by Lemma 2.1, (1) and (2) are also satisfied. This completes the proof of this Theorem.

Analogous results to Lemma 2.1, Lemma 2.2 and Theorem 2.3 can be proved in similar fashion for the loops satisfying the identical relations (4), (5), (6). We prove the following result for these loops.

THEOREM 2.6. *If* $(G,\cdot)$ *is RC-loop with* $\theta x\in N$, for all $x\in G$, *where* $\theta: G\to G$ *is a mapping such that* $x=\theta x\cdot\beta x$. *Then* $(G,\cdot)$ *satisfies* (4) *and hence* (5) *and* (6).

PROOF. From (iv) and the fact that $\theta x\in N$, we have

$$\begin{aligned}(yz\cdot x)\cdot\ \beta x&=(yz\cdot(\theta x\cdot\beta x))\cdot\beta x=((yz\cdot\theta x)\cdot\beta x)\cdot\beta x\\&=((y\cdot(z\cdot\theta x))\cdot\beta x)\cdot\beta x\\&=y\cdot[(z\cdot\theta x)\beta x\cdot\beta x]=y[(z\cdot(\theta x\cdot\beta x))\cdot\beta x]\\&=y\cdot(zx\cdot\beta x)\end{aligned}$$

which is (4). This completes the proof of this theorem.

REMARK 2.7. Let $(G,\cdot)$ be a loop. Then $(G,\cdot)$ is a group if and only if (1) or (2) or (3) holds with $\alpha x=1$.

REMARK 2.8. Let $(G,\cdot)$ be a loop. Then $(G,\cdot)$ is a group if and only if (4) or (5) or (6) holds with $\beta x=1$.

3. C-type

A loop $(G,\cdot)$ satisfying the identity

(vii) $$(yx\cdot x)z=y(x\cdot xz)$$

is called *C-loop* [2]. Let $\pi: G\to G$ be any mapping of G onto G. We consider the following identity in $(G,\cdot)$ similar to the above.

(9) $(yx\cdot\pi x)z=y(x\cdot(\pi x\cdot z))$. We have the following results regarding this identity.

THEOREM. 3.1. *If $(G,\cdot)$ is a loop in which (9) holds. Then $(G,\cdot)$ has inverse property.*

PROOF. Define ${}^{-1}x$ and x^{-1} in G as in (7) for all $x\in G$. Then for any x, y of G, given $\pi^2(x)$ and y in G, there exists u, $v\in G$ such that $\pi^2x\cdot u=y$ and $vx=y$. Using (9), we get

(10) $$\begin{aligned}{}^{-1}\pi x(\pi x\cdot y) &= {}^{-1}\pi x(\pi x\cdot(\pi^2x\cdot u))=(({}^{-1}\pi x\cdot\pi x)\cdot\pi^2x)\ u\\ &=\pi^2x\cdot u=y.\end{aligned}$$

As π is onto, this gives left inverse property and $y=\pi x$ in (10) gives with the help of (7), $u^{-1}={}^{-1}u$ and $(u^{-1})^{-1}=u$ for all $u\in G$. Also $(y\cdot\pi x)(\pi x)^{-1}=(vx\cdot\pi x)(\pi x)^{-1}=v(x\cdot(\pi x\cdot(\pi x)^{-1})=vx=y$. This gives right inverse property. Therefore $(G,\cdot)$ has inverse property.

THEOREM 3.2. *A loop $(G,\cdot)$ satisfies (9) if and only if $(R(x)R(\pi x),\ L(x)^{-1}L(\pi x)^{-1},\ I)$ is an autotopism of $(G,\cdot)$.*

PROOF. Replacing z by $(\pi x)^{-1}\cdot x^{-1}z$ in (9), we get using Theorem 3.1, $(yx\cdot\pi x)\ ((\pi x)^{-1}\cdot x^{-1}z)=y\ [x\cdot\pi x\cdot((\pi x)^{-1}\cdot x^{-1}z))]=y(x\cdot(x^{-1}z))=yz$. This gives $yR(x)R(\pi x)\cdot zL(x)^{-1}L(\pi x)^{-1}=(yz)\ I$, for all y, $z\in G$. Therefore, $(R(x)R(\pi x),\ L(x)^{-1}\ L(\pi x)^{-1},\ I)$ is an autotopism of $(G,\cdot)$. Taking $z=x\cdot(\pi x\cdot z)$, we get the converse.

THEOREM 3.3. *A loop $(G,\cdot)$ satisfies (9) if and only if $(G,\cdot)$ satisfies (1) (and hence (2), (3)) and (4) (and hence (5), (6)) with $\alpha x=\beta x=\pi x$, for all $x\in G$.*

PROOF. Suppose (9) holds in $(G,\cdot)$ Then we have the autotopism

(11) $$(R(x)R(\pi x),\ L(x)^{-1}\ L(\pi x)^{-1},\ I).$$

Using Theorem 3.1 and Theorem 1.1 (ii), we obtain the autotopism $(L(x)^{-1} L(\pi x)^{-1}, I, L(x)^{-1}L(\pi x)^{-1})$ of $(G, \cdot)$. Hence, $(L(x)^{-1}L(\pi x)^{-1}, I, L(x)^{-1}L(\pi x)^{-1})^{-1} = (L(\pi x)L(x), I, L(\pi x)L(x))$ is an autotopism of G, and Lemma 2.2 gives that (1) holds in $(G, \cdot)$. Again from (11), we have that $(I, R(x)R(\pi x), R(x) R(\pi x))$ is an autotopism of $(G, \cdot)$ and hence (4) is satisfied. Conversely, if (1) and (4) hold in $(G, \cdot)$, then $(G, \cdot)$ has inverse property. From Lemma 2.2 and Theorem 1.1 (ii), we get the autotopism $(R(x)R(\alpha x), L(x)^{-1} L(\alpha x)^{-1}, I)$ of $(G, \cdot)$ and hence (9) holds in $(G, \cdot)$.

THEOREM 3.4. *If* (9) *holds in* $(G, \cdot)$, *then* $\delta x \in N$ (*nucleus*) *for all* $x \in G$, *where* $\delta : G \to G$ *is a mapping such that* $\delta x = x \cdot \pi x$.

PROOF. If (9) holds, then G has inverse property. From Theorem 3.2 and Theorem 1.1 (ii). $(L(\delta x), I, L(\delta x))$ is an autotopism, showing thereby $\delta x \in N_\lambda$ (left nucleus) and hence from Theorem 1.1 (i), $\delta x \in N$ (nucleus).

REMARK 3.5. A loop $(G, \cdot)$ is a group if and only if (9) holds in $(G, \cdot)$ for π to be trivial mapping.

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