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A NOTE ON PURE STATES OF BANACH ALGEBRAS

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1. Introduction

J. Anderson [1] investigated the extension question for arbitrary C^*-algebras A and B. In this paper, we give a characterization of a pure state and properties of states on C^*-algebras. Moreover, we generalize some results in [1]. For instance, in §3 we show that if $G^+(f)$ commutes with every element of C^*-algebra and f is a pure state, then f is a homomorphism. In §4 we show that if f is a strictly pure state of a Banach*-algebra A and $L(f)\subset\alpha_f^{-1}(0)$, then f is a unique state extension of f_1 to A. Conversely, if f is the unique state extension of f_1 to A and dim $(A/L(f))<\infty$, then f is a strictly pure state, where f_1 is the restriction of f to $M(f)$.

2. Notations and preliminaries

By a *Banach*-algebra*, we mean a Banach algebra A with an involution * satisfying $\|a^*\|=\|a\|$ for all a in A. A positive linear functional f is called a *state* if $\|f\|=1$. An extreme state is called a *pure state* ([7]). For each state f on a Banach*-algebra A, let $L(f)=\{x\in A;\ f(x^*x)=0\}$ be the left ideal associated with f. Then f is a *strictly pure state* of A if f is a pure state and *-representation of A determined by f, π_f, is strictly irreducible on Hilbert space H_f ([2]). f is a strictly pure state of A if and only if $A/L(f)$ is complete in the norm $\|a+L(f)\|_2=f(a^*a)^{\frac{1}{2}}$ if and only if $\|\ \|_2$ and $\|\ \|_q$ are equivalent, where $\|\ \|_q$ is the quotient norm on $A/L(f)$ ([2]). If f is a pure state on a C^*-algebra, then f is a strictly pure state.

3. Pure states of C^*-algebras

PROPOSITION 3.1. *Suppose A is a *-algebra with an identity e. Let f be a linear functional on A with $f(e)=1$. Then $N(f)\subset L(f)$ if and only if f is a homomorphism, where $N(f)=\{x\in A;\ f(x)=0\}$.*

PROOF. Assume that $N(f)\subset L(f)$. Since $N(f)=\{x-f(x)e;\ x\in A\}$, $f(x^*x)=$

$f(x^*)f(x)$ for every $x \in A$. Then we are easy to see that $f(x^*y)=f(x^*)f(y)$ for all x, y in A. Thus f is a homomorphism.

Throughout this section, B shall always denote a C^*-algebra containing the identity e.

DEFINITION 3.2. For each state f on B, let
$M(f)=\{t \in B; f(tx)=f(xt)=f(t)f(x)$ for all $x \in B\}$ and
$G(f)=\{t \in B; |f(t)|=\|t\|=1\}$.
For each $x \in B$, let $\alpha_f(x)=\inf \{\|txt^*\|; t \in G(f)\}$.

R. V. Kadison proved in [3] that when f is a pure state of a C^*-algebra, then $N(f)=L(f)+L^*(f)$, where $L^*(f)=\{t \in B; t^* \in L(f)\}$. By proposition 3.1, a pure state on a C^*-algebra need not be a homomorphism. Hence we may consider the following proposition.

LEMMA 3.3. *Let f be a pure state on B. Assume that $G^+(f)=\{a \in G(f); a$ is a positive element$\}$ commutes with every element of B. Then $\alpha_f^{-1}(0)=L(f)$, where $\alpha_f^{-1}(0)=\{x \in B; \alpha_f(x)=0\}$.*

PROOF. In [1], $\alpha_f^{-1}(0)=L(f)+L^*(f)$ and $e \notin \alpha_f^{-1}(0)$. Thus $\alpha_f^{-1}(0)$ is closed proper subspace of B ([5]). Let $M=\{b+L(f); b \in \alpha_f^{-1}(0)\}$. Since $L(f) \subset \alpha_f^{-1}(0)$, M is $\| \|_q$-closed. Then M is $\| \|_2$-closed ([2]). By hypothesis, M is closed π_f-invariant proper subspace of $H_f=A/L(f)$. By ([7], 1. 21. 10), $\{\pi_f, H_f\}$ is irreducible. Thus $\alpha_f^{-1}(0)=L(f)$.

PROPOSITION 3.4. *Let f be a state on B. Suppose $G^+(f)$ commutes with every element of B. Then f is a pure state if and only if f is a homomorphism.*

PROOF. f is a pure state if and only if $\alpha_f(x-f(x)e)=0$ for all x in A ([1]). Lemma 3.3 shows that if f is a pure state, then $N(f)=L(f)$. Thus f is a homomorphism.

4. Extensions of states on Banach*-algebras

In this chapter, we carry some Anderson's results on C^*-algebra to Banach*-algebra. Throughout this section, A shall always denote a Banach*-algebra with an identity e. The previous notations $M(f)$, $G(f)$ and $\alpha_f(x)$ in C^*-algebra have the same meanings on A.

DEFINITION 4.1. Let f be a state on A. For each $x \in A$,
let $G'(f) = \{a \in G(f);\ a = a^* \text{ and } f(a) \geq 0\}$,
and $\beta_f(x) = \inf\{\|axa^*\|;\ a \in G'(f)\}$.

In C^*-algebra, $G^+(f)$ implies $G'(f)$.

The following proposition and $\alpha_f(x) = \beta_f(x)$ are followed by [1]. We observe that $\alpha_f^{-1}(0)$ is closed on A and $M(f)$ is Banach*-algebra. Also, $G(f) \subset M(f)$ and $G(f)$ is a topological semigroup on A.

PROPOSITION 4.2.

Let f be a pure state on A. If $L(f) \subset \alpha_f^{-1}(0)$, *then* $\alpha_f^{-1}(0) = \overline{L(f) + L^*(f)}$.

REMARK 4.3. (1) The condition $N(f) \subset \alpha_f^{-1}(0)$ in the proposition 4.2 is essential.

(2) Strictly pure state of A and $L(f) \subset \alpha_f^{-1}(0)$ are independent.

EXAMPLE.

Let C be a complex linear space and let A be the algebra of all matrices of the form $\begin{pmatrix} \alpha, & \beta \\ 0, & \alpha \end{pmatrix}$ with $\alpha \in C$, $\beta \in C$. Then it is well-known that A is a commutative Banach*-algebra, but A is not reduced. Define a complex mapping $f; A \longrightarrow C$ given by $f(x) = \alpha$, whenever $x = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$.

It is easy to see that f is a strictly pure state on A and

$$G(f) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix};\ a \in C \text{ and } |a| = 1 \right\}.$$

Thus $L(f) \not\subset \alpha_f^{-1}(0)$.

LEMMA 4.4. *Let f be a pure state on A with* dim $(A/L(f)) < \infty$. *Then f is a strictly pure state on A.*

PROOF. The identity map $a + L(f) \longrightarrow a + L(f)$ is a homeomorphism from $(A/L(f), \|\ \|_2)$ onto $(A/L(f), \|\ \|_q)$ ([4], p38). Thus $A/L(f)$ is complete in the norm $\|\ \|_2$. Therefore f is a strictly pure state on A.

PROPOSITION 4.5. *Let f be a state of A and let* f_1 *be the restriction of f to* $M(f)$. *If f is a strictly pure state and* $L(f) \subset \alpha_f^{-1}(0)$, *then f is the unique state extension of* f_1 *to A. Conversely, if f is the unique state extension of* f_1 *to*

A and dim $(A/L(f))<\infty$, *then f is a strictly pure state of A.*

PROOF. Suppose f is a strictly pure state on A and g is a state on A which agree with f on $M(f)$. By [2], $N(f)=\alpha_f^{-1}(0)$. Thus there exist $a_n \in G'(f)$ such that $\lim_{n\to\infty} a_n x a_n = \lim_{n\to\infty} f(x) a_n^2$ for all x in A. Therefore we have $f=g$ in A. Conversely, f is a pure state on A, because f_1 is a pure state on $M(f)$. By Lemma 4.4, f is a strictly pure state on A.

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