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ON DIRECT INJECTIVE MODULES

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1. Introduction

The concept of direct injectivity was introduced by W.K. Nicholson([3]). We know that injective modules are direct injective and its converse is not true in general. For instance, the Z-module Z_4 does not imply the direct injectivity, for the Z-module Z is not direct injective but quasi-injective.

In this paper, we investigate properties of direct injective modules. Throughout this paper M always denotes an R-module.

2. Results

DEFINITION 2.1. An R-module M is said to be *direct injective* if and only if given direct summand D of M with injection $1_D : D \longrightarrow M$ and a monomorphism $k : D \longrightarrow M$, there exists $f \in \mathrm{End}(M)$ such that $f \circ k = 1_D$.

PROPOSITION 2.2. *For an R-module M, M is direct injective if and only if for submodules A, B* of *M, B-direct summand of M and any monomorphism* $\Phi : M/B \longrightarrow A$, *there exists* $\Psi \in Hom_R(M, A)$ *with* $\Psi \circ \Phi = \nu$, *where* $\nu : M/B \longrightarrow M$ *is the canonical injection.*

PROOF. Assume that M is direct injective. Suppose B is a direct summand of M. Then for canonical injection $\nu : M/B \longrightarrow M$ and a monomorphism $\Phi : M/B \longrightarrow A$, there exists a homomorphism $g : M \longrightarrow M$ such that $g \circ i \circ \Phi = \nu$. Put $\Psi = g \circ i$, then Ψ is the required. The converse implication is quite obvious.

The proof of the following proposition is similar to that of proposition 2.3. in [4].

PROPOSITION 2.3. *A direct summand of a direct injective module is direct injective.*

PROPOSITION 2.4. *If* αM *is a direct summand of M for each* $\alpha \in End\ (M)$, *then M is direct injective.*

REMARK. The reverse of the above proposition 2.4. is not true in general.

Let $f: Z_4 \longrightarrow Z_4$ be a homomorphism such that $f(x)=\begin{cases} \bar{0} & \text{if } x=\bar{0},\ \bar{2}. \\ \bar{2} & \text{if } x=\bar{1},\ \bar{3}. \end{cases}$
Then $f(Z_4)=\{\bar{0},\ \bar{2}\}$ is not a direct summand of Z_4.

COROLLARY 2.5. *Every completely reducible module is direct injective.*

PROOF. M is completely reducible if and only if every submodule of M is a direct summand of M ([2]). It follows that a completely reducible module is direct injective.

COROLLARY 2.6. *If $End(M)$ is (von Neumann) regular, then M is direct injective.*

LEMMA 2.7. *Let $0 \longrightarrow L \xrightarrow{g} M \longrightarrow N \longrightarrow 0$ be a short exact sequence such that $L \oplus M$ is direct injective. Then this sequence splits.*

PROOF. Let $\nu_1 : L \longrightarrow L\oplus M$, $\nu_2 : M \longrightarrow L\oplus M$ be the corresponding canonical maps. By direct injectivity of $L\oplus M$, there exists a homomorphism $h \in End(L\oplus M)$ such that $\nu_1 = h\circ\nu_2\circ g$. Define a homomorphism $f(m)=(\pi_L\circ h\circ\nu_2)(m)$, where $\pi_L : L\oplus M \longrightarrow L$ is the corresponding projection map. Then $f\circ g = 1_L$ and hence the sequence splits.

PROPOSITION 2.8. *Let $\mu : K \longrightarrow M$ be a monomorphism such that M is injective. Then K is injective if and only if $K\oplus M$ is direct injective.*

PROOF. Assume that $K\oplus M$ is direct injective, then we have a short exact sequence; $0 \longrightarrow K \xrightarrow{\mu} M \longrightarrow M/\mathrm{Im}(\mu) \longrightarrow 0$. By Lemma 2.7, $M \cong K\oplus M/\mathrm{Im}(\mu)$.

COROLLARY 2.9. *The direct sum of two direct injective modules is direct injective if and only if every direct injective module is injective.*

PROOF. It is trivial because every module is isomorphic to a submodule of an injective module ([2]).

COROLLARY 2.10. *A ring R is completely reducible(=semisimple artinian) if and only if every R-module is direct injective.*

PROOF. By Proposition 2.8 and Corollary 2.9, it is trivial.

DEFINITION 2.11. A ring is called a *left* (right) *dc-ring* if every left (right) cyclic R-module is direct injective. A ring is called a *dc-ring* if it is a left and right *dc*-ring. Obviously every *pc*-ring is a *dc*-ring. A ring is said to be *self* *direct injective* if R is direct injective as an R-module. Trivially, any left *dc*-ring is self direct injective.

LEMMA 2.12. *Let R be a ring and I two sided ideal of R contained in the annihilator of M. Then M is direct injective over R if and only if it is direct injective over R/I.*

PROPOSITION 2.13. *A ring R is left dc if and only if R/A is left dc for each two sided ideal A of R.*

PROOF. Let R be a left *dc*-ring and A an ideal of R. Let I/A be any left ideal of R/A. Then, by [2], $(R/A)/(I/A)\cong R/I$ as an R-module. Since A annihilates the R-module R/I, we may consider R/I as R/A-module, Since R is a left *dc*-ring, R/I is R-direct injective. By Lemma 2.12, R/I, considered as an R/A-module, is R/A-direct injective. Hence any cyclic R/A-module is R/A -direct injective. i.e. R/A is a *duo*-ring.

PROPOSITION 2.14. *Every factor ring of a dc-ring R is self direct injective. Conversely if each factor ring of a dcθ-ring R is self direct injective, then R is a dc-ring.*

PROOF. Let A be an ideal of a *duo*-ring R. Then R/A is a *dc*-ring and hence self direct injective. Conversely, let M be a cyclic R-module. Then $M\cong R/A$ for some left ideal A of R. By hypothesis, R/A is R/A-direct injective. Hence, by Lemma 2.12, R/A is R-direct injective.

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