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ON STRUCTURES OF LEFT BIPOTENT NEAR RINGS

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1. Introduction

Throughout this paper N will mean a zero-symmetric near ring (that is, $a0=0$ for all $a\in N$) with or without identity, which satisfies the right distributive law. N is said to be *left bipotent* if $Na=Na^2$ for all $a\in N$, and an *s-near ring* if $a\in Na$ for all $a\in N$ ([6]). These types of near rings were introduced by J.L. Jat and S.C. Choudhary.

For convenience, we quote their results.

THEOREM 1.1. *If N is left bipotent, the following are equivalent.*

(1) *N is an s-near ring*

(2) *N has no non-zero nilpotent clements and*

(3) *N is regular (i.e. for any* $a\in N$*, there exists* $a'\in N$ *such that* $aa'a=a$*)*

On the other hand, Howard E. Bell proved in [7] that

THEOREM 1.2. *If N has no non-zero nilpotent elements, it is isomorphic to a subdirect sum of near rings with no non-zero divisors of zero.*

In this paper we investigated structures of left bipotent near rings and some elementary properties of near rings.

For undefined terminologies, we refer to [8].

2. Results

LEMMA 2.1. *Homomorphic images of left bipotent s-near rings are also such.*

PROOF. Let $f: N\longrightarrow N'$ be a homomorphism of near rings N onto N', and let N be a left bipotent s-near ring. If $a\in N'$, there exists $b\in N$ such that $f(b)=a$. By assumption, we have $Nb=Nb^2$. Then $f(Nb)=f(N)f(b)=N'a$ and $f(Nb^2)=f(N)f(b)^2=N'a^2$. Thus $N'a=N'a^2$. Now since $b\in Nb$, we have $a=f(b)\in f(Nb)=N'a$.

PROPOSITION 2.2. *Let N be a left bipotent s-near ring. Then it is isomorphic* *to a subdirect sum of near rings* $\{N_i | i\in I\}$, *where each* N_i *is a near field or has a right identity.*

PROOF. In view of Theorem 1.1, 1.2, and Lemma 2.1, we may assume that each N_i is a left bipotent s-near ring with no non-zero divisors of zero.

For each i, if N_i has a non-zero distributive element, then by [6] it is a near field. Now if N_i has no such an element, for any $0\neq r\in N_i$, there exists $r'\in N_i$ such that $rr'r=r$ by Theorem 1.1. Putting $e=rr'$, e is an idempotent. Then for each $x\in N_i$, we have $(xe-x)r=xer-xr=xr-xr=0$. Since N_i has no nonzero divisors of zero, we have $xe=x$. Hence e is a right identity for N_i.

We observe that homomorphic images of distributively generated (d.g.) near rings are d.g.. Thus a d.g. left bipotent s-near ring is isomorphic to a subdirect sum of near fields.

COROLLARY 2.3. *Let N be left bipotent with* 1. *N is isomorphic to a subdirect sum of near fields.*

PROOF. Since an epimorphism carries identity to identity, each subdirect summand of N endowed with an identity. Obviously an identity is a distributive element, so the proof is immediately established by Proposition 2.2.

COROLLARY 2.4. *A left bipotent near ring with* 1 *has commutative addition.*

PROOF. By Corollary 2.3. the proof is trivial.

PROPOSITION 2.5. *A left bipotent near ring with* 1 *is a ring iff it is d.g..*

PROOF. (⇒) Clear.

(⇐) With the aid of Corollary 2.4, we need only the left distributive law. Let $a, b, c\in N$ and put $a=a_1+\cdots+a_n$, where each a_i is distributive. Then we have

$$\begin{aligned} a(b+c) &= (a_1+\cdots+a_n)(b+c) \\ &= a_1(b+c)+\cdots+a_n(b+c) \\ &= a_1b+a_1c+\cdots+a_nb+a_nc \\ &= (a_1b+\cdots+a_nb)+(a_1c+\cdots+a_nc) \\ &= (a_1+\cdots+a_n)b+(a_1+\cdots+a_n)c \\ &= ab+ac. \end{aligned}$$

By a *left annihilator* of $r \in N$, we mean the set

$$l(r) = \{x \in N \mid xr = 0\}$$

It is immediate that $l(S) = \bigcap_{s \in S} l(s)$ for any non-empty subset S of N. Analogouly we may define right annihilators.

It is easily seen that left annihilators are left ideals and left annihilators of N-subgroups are ideals. But right annihilators are only closed under multiplications of elements of N on the right hand side.

Hereafter, for any non-empty subsets A and B of N,

$$AB = \{ab \mid a \in A \text{ and } b \in B\}$$

PROPOSITION 2.6. *If N has non-zero nilpotent elements, $l(S)$ is an ideal for every non-empty subset S of N.*

PROOF. We need only to show that $l(S)N \subset l(S)$. Let $x \in l(S)$ and $s \in S$. Then $xs = 0$ implies $(sx)^2 = s(xs)x = 0$. Hence $sx = 0$ by assumption. For any $r \in N$, $((xr)s)^2 = xr(sx)rs = 0$ implies $(xr)s = 0$. Thus $xr \in l(s)$. Hence $xr \in l(S) = \bigcap_{s \in S} l(s)$. Therefore $l(S)N \subset l(S)$.

In view of Theorem 1.1, we immediately have

COROLLARY 2.7. *In a left bipotent s-near ring, left annihilators are ideals.*

PROPOSITION 2.8. *Let B be a minimal N-subgroup of N, then either $B^2 = 0$ or there exists $e^2 = e \in B$ such that $B = Ne$.*

PROOF. If $B^2 \neq 0$, there exists $0 \neq b \in B$ such that $Bb \neq 0$. Since Bb is an N-subgroup and $0 \neq Bb \subset B$, $Bb = B$. Now let $l(b) = \{r \in N \mid rb = 0\}$ be the left annihilator of b in N. Then $l(b) \cap B = 0$.

Now $eb = b$ for some $0 \neq e \in B$, $e^2 b = eb$. Thus $(e^2 - e)b = 0$. Therefore $e^2 - e \in l(b) \cap B = 0$, so $e^2 = e$. Thus we have $B = Ne$ by the minimality of B.

Again by Theorem 1.1, we have

COROLLARY 2.9. *In a left bipotent s-near ring, every minimal N-subgroup of N has the form Ne for some $e^2 = e \in N$.*

LEMMA 2.10 *If N has* 1 *and every N-subgroup of N is finitely generated, then N has the maximum condition on N-subgroups.*

PROOF. Let $A_1 \subset A_2 \subset \cdots$ be a chain of N-subgroups of N. We set $A=\bigcup_{i=1}^{\infty} A_i$. Then A is an N-subgroup of N. Now let $\{a_1, \cdots, a_k\} \subset A$ be a generating set of A. Then $\{a_1, \cdots, a_k\} \subset A_n$ for some $n \in Z^+$. Hence $A \subset A_n \subset A$. Therefore $A_n = A_{n+1} = \cdots$ as required.

In a d.g. near ring, it is known that an N-subgroup of N is a left ideal iff it is a normal subgroup of $(N, +)$.

LEMMA 2.11. *Let N be d.g. and let A, B be left ideals of N, then $A+B$ is also a left ideal.*

PROOF. Since A and B are normal subgroup of $(N, +)$, we see that $A+B = B+A$ is also a normal subgroup of $(N, +)$. Now it suffices to show that $A+B$ is an N-subgroup of N.

Let $r \in N$. we may put $r = r_1 + \cdots + r_n$, where each r_i distributive or anti-distrbutive. For any element $a+b \in A+B$,

$$r(a+b) = (r_1 + \cdots + r_n)\ (a+b)$$
$$= r_1(a+b) + \cdots + r_n(a+b).$$

For each $1 \leq i \leq n$, if r_i is distributive, $r_i(a+b) = r_i a + r_i b \in A+B$, and if r_i is anti-distributive, $r_i(a+b) = r_i b + r_i a \in B+A = A+B$. In any way $r_i(a+b) \in A+B$ for $1 \leq i \leq n$. Therefore $r(a+b) \in A+B$ as required.

PROPOSITION 2.12. *Let N be d.g. with 1. If every additive subgroup of N is normal then N has the maximum condition on N-subgroups iff every N-subgroup of N is finitely generated.*

PROOF. ($\Leftarrow$) It is already shown in Lemma 2.10.

($\Rightarrow$) Under the hypothesis, the two concepts N-subgroup and left ideal coincide. So we turn the proof on left ideals.

Let A be a left ideal of N. Consider all finitely generated left ideals of N contained in A. By assumption, the set contains a maximal element, say, B.

Let $a \in A$ and consider $B+Na$. By assumption and Lemma 2.11, $B+Na$ is a left ideal and $B \subset B+Na \subset A$. Then $B = B+Na$ by the maximality of B. Hence $a \in B$ and consequently $A=B$.

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