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FURTHER RESULTS ON GENERALIZED CLOSED SETS IN TOPOLOGY

By William Dunham and Norman Levine

1. Introduction

Generalized closed (g-closed) sets in a topological space were introduced by Levine [5] in order to extend many of the important properties of closed sets to a larger family. For instance, it was shown that compactness, normality, and completeness in a uniform space are inherited by g-closed subsets. In the present paper, we continue the study of g-closed sets, obtaining characterizations in (2) and providing, in (3), examples of common topological structures which, although not necessarily closed, must be g-closed (e. g., derived sets, complete subspaces of uniform spaces, compact subsets and retracts of regular spaces). We prove a "generalized" Tietze Extension Theorem in (4) and apply this result, in theorem 5.3, to the problem of extending continuous, real-valued functions defined on compact subsets of completely regular spaces. Throughout the paper, many familiar results, and perhaps some unfamiliar ones, are derived as corollaries.

2. Characterizations of G-closed sets

DEFINITION 2.1. (Levine [5]) A subset A of a topological space is *g-closed* if $c(A)\subset O$ when $A\subset O$ and O is open. (Here "c" denotes the closure operator.)

THEOREM 2.2. *The following conditions are equivalent*:

(a) *A is g-closed*

(b) *for each $x\in c(A)$, $c(x)\cap A\neq\phi$*

(c) *$c(A)\setminus A$ contains no non-empty closed subsets*

PROOF. (a) implies (b): Suppose $x\in c(A)$ but $c(x)\cap A=\phi$. Then $A\subset\mathscr{C}c(x)$ (where $\mathscr{C}$ denotes the complement operator), and so $c(A)\subset\mathscr{C}c(x)$, contradicting $x\in c(A)$.

(b) implies (c): Let $F\subset c(A)\setminus A$ with F closed. If there is an $x\in F$, then, by (b), $\phi\neq c(x)\cap A\subset F\cap A\subset(c(A)\setminus A)\cap A$, a contradiction. We conclude that $F=\phi$.

(c) implies (a): If $A\subset O$ and O is open, then $c(A)\cap\mathscr{C}O$ is a closed subset of $c(A)\setminus A$ and thus is empty. Hence $c(A)\subset O$ and A is g-closed.

COROLLARY 2.3. *A is g-closed iff $A=F\setminus N$, where F is closed and N contains no non-empty closed subsets.*

PROOF. Necessity follows from theorem 2.2(c) with $F=c(A)$ and $N=c(A)\setminus A$. Conversely, if $A=F\setminus N$ and $A\subset O$ with O open, then $F\cap\mathscr{C}O$ is a closed subset of N and thus is empty. Hence $c(A)\subset F\subset O$.

COROLLARY 2.4. *In a T_1-space, g-closed sets are closed.*

PROOF. If A is g-closed in a T_1-space, theorem 2.2(c) implies $c(A)\setminus A=\phi$. Hence $c(A)=A$.

REMARK 2.5. A discussion of spaces in which the closed sets and the g-closed sets are identical—the so called $T_{\frac{1}{2}}$-spaces—can be found in Levine [5] and Dunham [1].

3. *G*-closed sets arising naturally in topology

LEMMA 3.1. *Let A be a subset of a topological space with A' its derived set, and suppose $A'\subset O$ for O open. Then $A''\subset O$.*

PROOF. Suppose $x\in A''$ but $x\notin O$. Then $x\notin A'$ and so, for some open set U, $x\in U$ and $A\cap U\subset\{x\}$. But $x\in A''$ implies $y\in A'\cap U\cap\mathscr{C}\{x\}$ for some y. Now, $y\in O\cap U$ and $y\in A'$ and so $\phi\neq A\cap O\cap U\cap\mathscr{C}\{y\}\subset A\cap U\subset\{x\}$. It follows that $x\in O$, a contradiction.

THEOREM 3.2. *In any topological space, derived sets are g-closed.*

PROOF. If A is any subset of a topological space with $A'\subset O$ for O open, the previous lemma implies $c(A')=A'\cup A''\subset O$.

COROLLARY 3.3. *Derived sets in a compact space are compact.*

PROOF. By the previous result, derived sets are g-closed, and, in [5], theorem 3.1, Levine has shown that g-closed subsets of a compact space are compact.

REMARK 3.4. A space X is said to be *weakly Hausdorff* if $c(x)=c(y)$ whenever there is a net $S: D\longrightarrow X$ with $\lim S=x$ and $\lim S=y$. Of primary importance is the fact that any regular space or any Hausdorff space is weakly Hausdorff (see Dunham [2] for details). We shall use this idea in the next four examples of g-closed sets.

THEOREM 3.5. *If A is a compact subset of a weakly Hausdorff space, then*

A is g-closed.

PROOF. For $x \in c(A)$, there is a net $S: D \longrightarrow A$ with $\lim S = x$ and, by compactness, there is a subnet $T: E \longrightarrow A$ with $\lim T = a$ for some $a \in A$. The weakly Hausdorff property implies $c(a) = c(x)$ and thus $a \in c(x) \cap A$. By theorem 2.2(b), A is g-closed.

COROLLARY 3.6. *A compact subset of a regular space is g-closed and a compact subset of a Hausdorff space is closed.*

PROOF. Use corollary 2.4, remark 3.4, and the previous result.

THEOREM 3.7. *If A is a retract of a weakly Hausdorff space X, then A is g-closed.*

PROOF. Let $r: X \longrightarrow A$ be the retraction and let $x \in c(A)$. Then there is a net $S: D \longrightarrow A$ with $\lim S = x$, and it follows that $\lim S = \lim r \circ S = r(x)$. We conclude that $c(r(x)) = c(x)$ and thus $c(x) \cap A \neq \phi$.

COROLLARY 3.8. *A retract of a regular space is g-closed and a retract of a Hausdorff space is closed.*

THEOREM 3.9. *Let $f: X \longrightarrow Y$ be continuous, with Y a weakly Hausdorff space, and let $G_f = \{(x, f(x)) : x \in X\}$ be the graph of f. Then G_f is g-closed in $X \times Y$.*

PROOF. For $(x, y) \in c(G_f)$, there is a net $S: D \longrightarrow G_f$, denoted $S(d) = (x_d, f(x_d))$, with $\lim S = (x, y)$. Then, by continuity, $f(x) = \lim f(x_d) = y$, and so $c(f(x)) = c(y)$. Hence $(x, f(x)) \in G_f \cap (c(x) \times c(y)) = G_f \cap c(\{(x, y)\})$, and G_f is g-closed by theorem 2.2(b).

COROLLARY 3.10. *The graph of a continuous function whose range lies in a regular space is g-closed. In particular, the diagonal of a regular space is g-closed.*

THEOREM 3.11. *Suppose $(X, \mathscr{U})$ is a uniform space with $A \subset X$ a complete subspace. Then A is g-closed in the uniform topology.*

PROOF. For $x \in c(A)$ there is a net $S: D \longrightarrow A$ with $\lim S = x$. Then S is A-Cauchy, and so $\lim S = a$ for some $a \in A$. Since X is completely regular, it is weakly Hausdorff by remark 3.4. Hence $a \in c(x) \cap A$ and A is g-closed.

REMARK 3.12. By the previous result and Levine [5], theorem 3.4, we see that complete subspaces of uniform (or pseudometric) spaces are g-closed, while g-closed subspaces of complete uniform (or complete pseudometric) spaces are complete. As an immediate consequence we have the familiar:

COROLLARY 3.13. *Complete subspaces of separated uniform spaces or of metric spaces are closed.*

THEOREM 3.14. *Let (Y, d) be a pseudometric space and let $B(X, Y)$ be the family of bounded maps from X to Y with $\sigma(f, g)=\sup\{d(f(x), g(x)): x\in X\}$ the pseudometric of uniform convergence on $B(X, Y)$. Further, let $\beta: Y\longrightarrow B(X, Y)$ be the natural embedding given by $\beta(y)(x)=y$ for all $x\in X$. Then $\beta[Y]$ is g-closed in $B(X, Y)$ with the pseudometric topology.*

PROOF. If $f\in c(\beta[Y])$, then, for each natural number n, there is a $y_n\in Y$ with $\sigma(f, \beta(y_n))<1/n$. Fixing $x_0\in X$, we assert that $\beta(f(x_0))\in c(f)\cap\beta[Y]$ and it suffices to show $\sigma(f, \beta(f(x_0)))=0$. But, for any $x\in X$ and for n arbitrary, we have $d(f(x), \beta(f(x_0))(x))=d(f(x), f(x_0))<d(f(x), y_n)+d(y_n, f(x_0))<\sigma(f, \beta(y_n)) +\sigma(\beta(y_n), f)<2/n$. Thus, $\sigma(\beta(f(x_0)), f)=0$ and $\beta[Y]$ is g-closed by theorem 2.2 (b).

COROLLARY 3.15. *(Y, d) is complete iff $(B(X,Y),\sigma)$ is complete.*

PROOF. Necessity is a standard result, and sufficiency follows by combining theorem 3.14 and remark 3.12 and noting that β is an isometry.

4. A generalized Tietze extension theorem

REMARK 4.1. In this section we shall prove that "closed" can be replaced by "g-closed" in the statement of the Tietze Extension Theorem. We begin by recalling a theorem of A.D. Taimanov:

THEOREM 4.2. *If $A\subset X$ and $f: A\longrightarrow Y$ is continuous, where Y is a compact, Hausdorff space, then the following are equivalent:*

(a) *f has a continuous extension to $c(A)$*

(b) *for every G_1 and G_2, closed and disjoint in Y, the closures of $f^{-1}[G_1]$ and $f^{-1}[G_2]$ are disjoint in X.*

PROOF. See Taimanov [7] (in Russian) or Engelking [3], theorem 3.2.1.

THEOREM 4.3. *If $A\subset X$ is g-closed and $f: A\longrightarrow Y$ is continuous, where Y is compact and Hausdorff, then there exists a continuous $F: c(A)\longrightarrow Y$ with $F|_A=f$.*

PROOF. Let G_1 and G_2 be closed, disjoint subsets of Y. Defining $D=c(A)\cap c(f^{-1}[G_1])\cap c(f^{-1}[G_2])$, we assert that $D\subset\mathscr{C}A$. For, if $x\in D\cap A$, then for $i=1,2$, we have $x\in A\cap c(f^{-1}[G_i])=c_A(f^{-1}[G_i])=f^{-1}[G_i]$ by continuity, and thus $f(x)\in G_1\cap G_2$, a contradiction. Hence D is an X-closed subset of $c(A)\setminus A$ and so $D=\phi$ by theorem 2.2(c). The continuous extension of f to $c(A)$ follows from theorem 4.2.

COROLLARY 4.4. *The previous result holds if "compact" is replaced by "locally compact".*

PROOF. If Y is locally compact and Hausdorff, we let $Y^*=Y\cup\{\infty\}$ be the one-point compactification of Y. Then Y^* is a compact, Hausdorff space and so there is a continuous $F: c(A)\longrightarrow Y^*$ with $F|_A=f$. But $F^{-1}[\{\infty\}]$ is a closed subset of $c(A)\setminus A$ and thus is empty. Hence $F: c(A)\longrightarrow Y$ is the desired extension.

THEOREM 4.5. (*Generalized Tietze Extension Theorem*) *A continuous, real-valued function defined on a g-closed subset of a normal space has a continuous extension to the entire space.*

PROOF. If A is a g-closed subset of the normal space X and $f: A\longrightarrow R$ is continuous, then there is a continuous $F: c(A)\longrightarrow R$ with $F|_A=f$ by corollary 4.4. The Tietze Extension Theorem then provides a continuous $F^*: X\longrightarrow R$ with $F^*|_{c(A)}=F$. Thus $F^*|_A=f$.

COROLLARY 4.6. *A continuous, real-valued function defined on a complete subspace of a pseudometic space has a continuous extension to the entire space.*

PROOF. A pseudometric space is normal and a complete subspace is g-closed by remark 3.12.

REMARK 4.7. "Pseudometric" can not be replaced by "uniform" in the previous result. Let Δ be an uncountable set and, for each $\alpha\in\Delta$, let $(X_\alpha, \mathscr{U}_\alpha)$ be the reals with the usual uniformity. Then $(X, \mathscr{U})=\times\{(X_\alpha, \mathscr{U}_\alpha): \alpha\in\Delta\}$ is a complete uniform space. By Stone [6], X with the uniform topology is not normal, and so there is a closed (and thus complete) subspace A of X and a continuous $f: A\longrightarrow R$ which can not be extended continuously to all of X.

5. An application

REMARK 5.1. We conclude this paper by applying the concepts developed above to the problem of extending continuous, real-valued functions from compact subsets of a topological space to the space itself. We shall use the following characterization of complete regularity, which is the non-T_1 analogue of the well-known result that a space is Tychonoff (i. e., completely regular and T_1) iff it is homeomorphic to a subspace of a compact, Hausdorff space:

THEOREM 5.2. *A space is completely regular iff it is homeomorphic to a subspace of a compact, regular space.*

PROOF. See Dunham [2], corollary 7.8.

THEOREM 5.3. *A continuous, real-valued function defined on a compact subset of a completely regular space has a continuous extension to the entire space.*

PROOF Let A be a compact subset of the completely regular space X and let $f: A \longrightarrow R$ be continuous. By theorem 5.2, there is a compact, regular space X^* and an $h: X \longrightarrow X^*$ so that $h: X \longrightarrow h[X]$ is a homeomorphism. We note that:

(i) X^* is compact and regular and thus is normal and regular.

(ii) $h[A]$ is compact in X^* and thus is g-closed in X^* by corollary 3.6.

(iii) $h|_A: A \longrightarrow h[A]$ is a homeomorphism and so $f \circ (h|_A)^{-1}: h[A] \longrightarrow R$ is continuous.

By (i)—(iii) and theorem 4.5, there is a continuous $F^*: X^* \longrightarrow R$ with $F^*|_{h[A]} = f \circ (h|_A)^{-1}$. Define $F: X \longrightarrow R$ by $F = F^* \circ h$. Then F is continuous, real-valued, and, for $x \in A$, $F(x) = F^*(h(x)) = f(x)$. Thus F is the desired extension of f.

COROLLARY 5.4. *A continuous, real-valued function defined on a compact subset of a uniform (or regular, normal; or regular paracompact; or regular, second axiom) space has a continuous extension to the entire space.*

PROOF. All such spaces are completely regular.

REMARK 5.5. In theorem 5.3, "completely regular" can not be weakened to "regular". Hewitt [4] provides an example of a regular, T_1 space on which the only continuous, real-valued functions are constant. Thus, any non-constant, real-valued function defined on a two-point subspace is continuous but has no continuous extension to the entire space.

William Dunham
Department of Mathematics
Hanover College
Hanover, Indiana 47243
U.S.A.

and

Norman Levine
Department of Mathematics
The Ohio State University
Columbus, Ohio 43210
U.S.A.

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