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QUASI ORDERED BITOPOLOGICAL SPACES II

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The study of bitopological spaces arose when Kelly ([3]) started investigating a certain nonsymmetric generalization of metric spaces. These spaces are named quasimetric spaces by Kelly. Every quasimetric, as noted by Kelly, led to the consideration of a conjugate quasimetric, this resulting in the study of the structure of bitopological spaces. These spaces, i. e., a set endowed with a quasimetric and its conjugate, as demonstrated by Kelly, satisfy many interesting pairwise separation properties analogous to those satisfied by metric spaces. Thus Kelly found that the problem of quasimetrizability of a topological space (X, τ) leads to the consideration of yet another topology τ' on X whose behaviour with τ is in many ways natural. In a recent paper, Raghavan and Reilly ([11]) strongly expressed the view that the study of any nonsymmetric structure which induces a topology on a set naturally leads to the consideration of a bitopological space.

The present paper treats the study of ordered bitopological spaces. In fact, in earlier papers ([9], [10]) the author made a systematic study of bitopological spaces equipped with an order relation while the present paper is a sequel to these papers.

In the sections two and three, pairwise G compact, pairwise G Lindelöf, pairwise G countably compact spaces are studied. In the fourth section we introduce what is called the notion of *induced orders* and demonstrate that every T_0 space is weakly orderable in a certain sense made precise in that section. Also we show that if a topology is compatible with a quasiuniformity as also with its conjugate, the space is orderable and locally convex. In the fifth section we study pairwise G perfectly normal spaces and their properties. As is usually the case, many well-known results in unitopological spaces can be obtained as particular cases by taking the graph G of the order relation to be equal to Δ, the graph of the discrete order relation and setting the two topologies to be one and the same.

1. Preliminaries and notation.

By a *quasi ordered set* we mean a set endowed with a relation which is reflexive and transitive. If the relation is further anti-symmetric, the set is said to be *partially ordered*. Let X be a set endowed with a partial order $\leqq$. A subset A of X is said to be *decreasing* if and only if $a\leqq b$ and $b\in A\Rightarrow a\in A$. The concept of an *increasing* set is defined dually. In fact the concepts of decreasing and increasing sets can be defined even in quasi ordered sets. The following are the abbreviations and notations used in the text.

dec. =decreasing

inc. =increasing

b. t. s. =bitopological space

u. s. c. =upper semi-continuous

l. s. c. =lower semi-continuous

$\tau_i$cl(A)=The closure of A in the topology τ_i.

nbd=neighbourhood

$L(A)=\{y : y\leqq x$ for some $x\in A\}$

$M(A)=\{y : x\leqq y$ for some $x\in A\}$

$H_i^l(A)$=The smallest dec. τ_i closed set containing A.

$H_i^m(A)$=The smallest inc. τ_i closed set containg A.

$A\ {}_2{<}_1 B \Leftrightarrow$There exists a dec. τ_2 open set $U\supset A$ and an inc. τ_1 open set $V\supset B$ such that $U\cap V=\phi$.

N=The set of natural numbers.

It is easy to see that $L(A)$ is the intersection of all dec. subsets of X which contain A. Thus it is the smallest dec. set containing A. Dually $M(A)$ is the intersection of all inc. subsets of X which contain A. The complement of an inc. (dec.) set is dec. (inc.). A realvalued function f on a quasiordered set is called *monotone* (*antitone*) if $x\leqq y\Rightarrow f(x)\leqq f(y)(f(x)\geqq f(y))$.

2. Pairwise G compact spaces

The following definition of pairwise compactness is due to Kim [4].

DEFINITION 2.1. Let (X, τ_1, τ_2) be a *b. t. s.* Let $\tau(i, V)=\{\phi, X, \{U\cup V\,|\,U\in\tau_i\}\}$ where $V\in\tau_j(i, j=1, 2; i\neq j)$. If $\tau(i, V)$ is compact for every $V\in\tau_j$, hen the space is called *pairwise compact*.

DEFINITION 2.2. Let (X, τ_1, τ_2) be a b. t. s. equipped with a partial order $\leqq$. The partial order is said to be *weakly continuous* if and only if the graph G of the partial order is closed in the product topology $\tau_1 \times \tau_2$.

DEFINITION 2.3. A pairwise compact b. t. s. (X, τ_1, τ_2) equipped with a weakly continuous partial order is called *pairwise G compact.*

We have proved the following result in [9, Theorem 2.7].

LEMMA 2.4. *If (X, τ_1, τ_2) is a b. t. s. equipped with a weakly continuous partial order, then if K is $\tau_2(\tau_1)$ compact then $L(K)$ $(M(K))$ is $\tau_1(\tau_2)$ closed.*

LEMMA 2.5. *Let (X, τ_1, τ_2) be a pairwise G compact b. t. s. If P is a dec. (inc.) set and V is a τ_2 nbd (τ_1 nbd) of P, then there exists a dec. τ_2 open (inc. τ_1 open) set U such that $P \subset U \subset V$.*

PROOF. Since τ_2 closed subset of a pairwise compact space is τ_1 compact [4, Theorem 2.9], $\tau_2\mathrm{cl}(X-V)$ is τ_1 compact so that $M(\tau_2\mathrm{cl}(X-V))$ is τ_2 closed by Lemma 2.4 above. If we write $U=X-M(\tau_2\mathrm{cl}(X-V))$ then U is dec. τ_2 open set such that $U \subset V$. Also, if $x \notin U$, then $x \in M(\tau_2\mathrm{cl}(X-V))$, so that there exists $y \in \tau_2\mathrm{cl}(X-V)$ with $y \leqq x$. Hence it follows that there exists $y \leqq x$ with $y \notin P$. Since P is dec. $x \notin P$. Therefore $P \subset U$.

LEMMA 2.6. *Let (X, τ_1, τ_2) be a pairwise G compact b. t. s. If F_1 and F_2 are τ_1 and τ_2 closed subsets of X respectively such that $x \nleqq y$ for $x \in F_2$ and $y \in F_1$, then there exists an inc. τ_1 nbd U of F_2 and a dec. τ_2 nbd V of F_1 such that $U \cap V = \phi$.*

PROOF. Since the partial order is weakly continuous, there is an inc. τ_1 nbd P of x and a dec. τ_2 nbd Q of y such that $P \cap Q = \phi$ whenever $x \nleqq y$ by [9, Proposition 2.2]. This is particular case when $x \in F_2$ and $y \in F_1$. Let $y \in F_1$. Since F_2 is τ_1 compact, there exist inc. τ_1 nbd U_i of $x_i \in F_2$ and dec. τ_2 nbd V_i of y such that $V_i \cap U_i = \phi$ $(i=1, 2, 3, \cdots, n)$. If $U_y = \cup \{U_i \mid i=1, 2, 3, \cdots, n\}$ and $V_y = \cap \{V_i \mid 1, 2, 3, \cdots, n\}$, then U_y is an inc. τ_1 nbd of F_2, V_y is a dec. τ_2 nbd of y and $U_y \cap V_y = \phi$. Again there exist a finite number of points $y_j \in F_1$, a dec. τ_2 nbd V_{y_j} of y_j and inc. τ_1 nbd U_{y_j} of F_2 such that $V_{y_j} \cap U_{y_j} = \phi$ $(j=1, 2, 3, \cdots, m)$ and $\cup \{V_{y_j} \mid j=1, 2, 3, \cdots, m\} = V$ is a dec. τ_2 nbd of F_1 (since F_1 is τ_2 compact). If we set $U = \cap \{U_{y_j} \mid j=1, 2, 3, \cdots, m\}$, then U is an inc. τ_1 nbd of F_2 and $U \cap V = \phi$.

THEOREM 2.7. *A pairwise G compact space is pairwise normally ordered.*

PROOF. Recall that [9, Definition 3.1] a b. t. s. (X, τ_1, τ_2) equipped with an order relation is pairwise normally ordered if A and B are subsets of X which are dec. τ_1 closed and inc. τ_2 closed respectively and disjoint, then $A_2 <_1 B$. Now the result follows from Lemmas 2.5 and 2.6.

DEFINITION 2.8. Let (X, τ_1, τ_2) be a b. t. s. Let $\tau(i, V)=\{\phi, X, \{U\cup V \mid U\in\tau_i\}\}$ where $V\in\tau_j (i, j=1, 2; i\neq j)$. If $\tau(i, V)$ is Lindelöf for every $V\in\tau_j$, then the space is called *pairwise Lindelöf.*

LEMMA 2.9. *Let (X, τ_1, τ_2) be a pairwise Lindelöf b. t. s. Then if A is τ_j closed then A is τ_i Lindelöf.*

PROOF. The proof of this Lemma runs analogous to that of [4, Theorem 2.9].

THEOREM 2.10. *A pairwise regular pairwise Lindelöf b. t. s. is pairwise normal.*

PROOF. Let A and B be τ_1 and τ_2 closed sets respectively which are disjoint. Since the space is pairwise regular, for each point $a\in A$, there exists a τ_2 open set U_a such that $\tau_1\mathrm{cl}(U_a)$ does not meet B. Similarly for each point $b\in B$, there exists τ_1 open set V_b such that $\tau_2\mathrm{cl}(V_b)$ does not meet A. Since A is τ_2 Lindelöf, we have a sequence of τ_2 open sets $\{U_n \mid n\in N\}$ covering A. This is a subcovering of $\{U_a \mid a\in A\}$. Similarly since the set B is τ_1 Lindelöf, we have a sequence of τ_1 open sets $\{V_n \mid n\in N\}$ covering B. This is again a subcovering of $\{V_b \mid b\in B\}$. Now set $U'_n=U_n-\cup\{\tau_2\mathrm{cl}(V_q) \mid q\leqq n\}$, $V'_m=V_m-\cup\{\tau_1\mathrm{cl}(U_p) \mid p\leqq m\}$. Take $U=\cup\{U'_n \mid n\in N\}$ and $V=\cup\{V'_m \mid m\in N\}$. Then $U\supset A$, $V\supset B$, $U\cap V=\phi$ and U and V are τ_2 and τ_1 open respectively, so that the space is pairwise normal.

DEFINITION 2.11. A b. t. s. (X, τ_1, τ_2) equipped with a quasiorder whose graph is G is called *pairwise G regular* if and only if given dec. τ_1 closed set A (a point $x\notin B$) and a point $x\notin A$ (inc. τ_2 closed set B) there exist sets U and V such that $A\subset U$ $(x\in U)$ and $x\in V (B\subset V)$, U and V are dec. τ_2 open and inc. τ_1 open sets respectively and $U\cap V=\phi$.

THEOREM 2.12. *Every pairwise Lindelöf pairwise G regular space is pairwise normally ordered.*

PROOF. Let A and B be dec. τ_1 closed and inc. τ_2 closed sets such that $A\cap B=\phi$. Since the space is pairwise G regular, for each point $a\in A$, there exists a dec. τ_2 open set U_a such that $H_1^l(U_a)\cap B=\phi$. Similarly for each $b\in B$, there exists an inc. τ_1 open set V_b such that $H_2^m(V_b)\cap A=\phi$. Since A and B are respectively τ_2 and τ_1 Lindelöf, there is a subcovering $\{U_n|n\in N\}$ of $\{U_a|a\in A\}$ and $\{V_n|n\in N\}$ of $\{V_b|b\in B\}$. Set $U'_p=U_p-U\{H_2^m(V_r)|r\leqq p\}$ and $V'_q=V_q-\cup\{H_1^l(U_r)|r\leqq q\}$. Again set $U=\cup\{U'_p|p\in N\}$, $V=\cup\{V'_q|q\in N\}$. U and V are dec. τ_2 open and inc. τ_1 open sets such that $U\supset A$, $V\supset B$ and $U\cap V=\phi$. Hence the space is pairwise normally ordered.

The notion of a P-space is well-known. A topological space is called a *P-space* if and only if every G_δ set is open. A b. t. s. (X, τ_1, τ_2) is called a *bitopological (P) space* if and only if (X,τ_1) and (X,τ_2) are both P-spaces. Let us abbreviate bitopological (P) space as b. t. (P) space.

Analogous to [9, Theorem 2.7] the following proposition can be proved.

PROPOSITION 2.13. *Let (X, τ_1, τ_2) be a b. t. (P) space equipped with a weakly continuous partial order. If K is $\tau_1(\tau_2)$ Lindelöf, then $M(K)$ $(L(K))$ is $\tau_2(\tau_1)$ closed.*

Now let us define pairwise G Lindelöf bitopological spaces.

DEFINITION 2.14. A b. t. (P) space (X, τ_1, τ_2) equipped with a partial order whose graph is G is called *pairwise G Lindelöf* if and only if the space is pairwise Lindelöf and G is weakly continuous.

In the same way as we proved 2.5, 2.6 and 2.7, the following results also can be proved.

PROPOSITION 2.15. *Let X be a pairwise G Lindelöf b. t. (P) space. If Q is a dec. (inc.) subset of X and V is a $\tau_2(\tau_1)$ nbd of Q, then there exists a dec. τ_2 open (inc. τ_1 open) set U such that $Q\subset U\subset V$.*

PROPOSITION 2.16. *Let X be a pairwise G Lindelöf b. t. (P) space. If F_1 and F_2 are τ_1 and τ_2 closed subsets of X respectively such that $x\not\leqq y$ for $x\in F_2$ and $y\in F_1$. Then there exists an inc. τ_1 nbd U and a dec. τ_2 nbd V such that $F_2\subset U$, $F_1\subset V$ and $U\cap V=\phi$.*

PROPOSITION 2.17. *A pairwise G Lindelöf b. t. (P) space is pairwise norm-*

ally ordered.

3. Further characterizations.

DEFINITION 3.1. Let (X, τ_1, τ_2) be a b.t.s. equipped with a partial order whose graph is G. The space is called *pairwise* $G\alpha$ if and only if for each $x\in X$, $L(x)=\bigcap\{V_i|i\in N\}$ and $M(x)=\bigcap\{W_i|i\in N\}$ where each V_i is a dec. τ_2 open set and each W_i is an inc. τ_1 open set.

PROPOSITION 3.2. *If* (X, τ_1, τ_2) *is a b. t. s. which is pairwise* $G\alpha$, *then* $L(x)$ *is* τ_1 *closed and* $M(x)$ *is* τ_2 *closed.*

PROOF. Suppose $y\notin L(x)$. Then $y\nleqq x$ so that $M(y)\cap L(x)=\phi$. Hence $x\notin M(y)$. Thus there exists an inc. τ_1 open set W such that $x\notin W$ and $W\supset M(y)$. In other words, $y\in W$ and $W\cap L(x)=\phi$ so that $L(x)$ is τ_1 closed. Similarly one can prove that $M(x)$ is τ_2 closed.

DEFINITION 3.3. Let (X,τ_1,τ_2) be a b.t.s. equipped with a partial order whose graph is G. The space is called *pairwise* $G\beta$ if and only if for each $x\in X$, $L(x)=\bigcap\{H_1^l(V_i)|i\in N\}$ and $M(x)=\bigcap\{H_2^m(W_i)|i\in N\}$ where each V_i is a dec. τ_2 open set containing x and each W_i is inc. τ_1 open set containing x.

It is to be noted that when $\tau_1=\tau_2$ and $G=\varDelta$, the discrete order, the Definitions 3.1 and 3.3 reduce to the definitions of E_0 and E_1 spaces of Aull [1]. Also let us call a pairwise $G\beta$ space to be pairwise E_1 whenever $G=\varDelta$.

PROPOSITION 3.4. *Let* (X, τ_1, τ_2) *be a b. t. s. equipped with a partial order whose graph is* G. *Consider the following statements.*

(i) X *is pairwise* $G\alpha$.

(ii) X *is pairwise* $G\beta$.

(iii) G *is weakly continuous.*

(iv) $L(x)$ *and* $M(x)$ *are* τ_1 *and* τ_2 *closed subsets of* X *respectively for each* $x\in X$.

Then the following implications hold.

$$\begin{array}{ccc}(ii) & \Rightarrow & (i)\\ \Downarrow & & \Downarrow\\ (iii) & \Rightarrow & (iv).\end{array}$$

PROOF. That (i)$\Rightarrow$(iv) is precisely the Proposition 3.2 above.
That (iii)$\Rightarrow$(iv) is proved in [9, Proposition 2.3].

(ii)$\Rightarrow$(i). Let $\{V_i|i\in N\}$ be a sequence of dec. τ_2 open sets containing x such that $L(x)=\cap\{H_1^l(V_i)|i\in N\}$. If $y\in L(x)$, then $y\leqq x\in V_i$ for each i. Since V_i is decreasing, $y\in V_i$ for each i so that $L(x)\subset\cap\{V_i|i\in N\}$. Also if $y\in\cap\{V_i|i\in N\}$, then $y\in V_i\subset H_1^l(V_i)$ for each i so that $y\in\cap\{H_1^l(V_i)|i\in N\}=L(x)$. Therefore $L(x)=\cap\{V_i|i\in N\}$. Similarly $M(x)=\cap\{W_i|i\in N\}$ where each W_i is an inc. τ_1 open set.

(ii)$\Rightarrow$(iii). If $x\not\leqq y$, then $y\notin L(x)$ so that there exists a dec. τ_2 open set V such that $x\in V$ and $y\notin H_1^l(V)$. Let $U=X-H_1^l(V)$. Clearly V and U are dec. τ_2 open set and inc. τ_1 open set respectively such that $x\in V$, $y\in U$ and $U\cap V=\phi$. Hence by [9, Proposition 2.2] G is weakly continuous.

PROPOSITION 3.5. *If (X,τ_1,τ_2) is pairwise G regular then, in Proposition* 3.4, (i)$\Leftrightarrow$(ii) *and* (iii)$\Leftrightarrow$(iv).

PROOF. (i)$\Rightarrow$(ii). Let $\{V_i|i\in N\}$ be a countable collection of dec. τ_2 open sets such that $L(x)=\cap\{V_i|i\in N\}$. Now $y\notin L(x)$

$\Rightarrow y\notin V_n$ for some $n\in N$ and $x\notin M(y)$.

$\Rightarrow V_n\cap M(y)=\phi$.

$\Rightarrow$(Since the space is pairwise G regular), there exists disjoint sets W_n and U_n such that $x\in W_n$, $M(y)\subset U_n$, W_n and U_n are dec. τ_2 open and inc. τ_1 open respectively.

$\Rightarrow x\in P_n\subset X-U_n$ where $P_n=W_n\cap V_n$ and $y\notin X-U_n$ which is dec. τ_1 closed.

$\Rightarrow x\in L(x)\subset P_n\subset H_1^l(P_n)\subset X-U_n$ and $y\notin H_1^l(P_n)$ for some $n\in N$.

$\Rightarrow y\notin\cap\{H_1^l(P_i)|i\in N\}$.

Also, $L(x)\subset H_1^l(P_i)$ for each i, so that $L(x)=\cap\{H_1^l(P_i)|i\in N\}$. Similarly we can prove the existence of a countable collection $\{Q_i|i\in N\}$ of inc. τ_1 open subsets of X such that $M(x)=\cap\{H_2^m(Q_i)|i\in N\}$.

That (iv)$\Rightarrow$(iii) follows easily on using the fact that the space is pairwise G regular.

PROPOSITION 3.6. *Let (X,τ_1,τ_2) be a pairwise $G\beta$ b. t. s. If K is $\tau_2(\tau_1)$ countably compact, then $L(K)(M(K))$ is $\tau_1(\tau_2)$ closed.*

PROOF. Let $y\in X-L(K)$. Since the space is pairwise $G\beta$ there exists a countable collection $\{Q_i | i \in N\}$ of inc. τ_1 open sets each containing y such that $M(y) = \cap \{H_2^m(Q_i) | i \in N\}$. Since $M(y) \cap L(K) = \phi$, $K \subset L(K) \subset X - M(y) \subset \cup \{X - H_2^m(Q_i) | i \in N\}$. Hence there exists a finite number of integers $n_1, n_2, \cdots, n_r$ such that $K \subset \cup \{X - H_2^m(Q_{n_j}) | j = 1, 2, 3, \cdots, r\}$; (notice that K is τ_2 contably compact). Since each $X - H_2^m(Q_{n_j})$ is dec., $K \subset L(K) \subset \cup \{X - H_2^m(Q_{n_j}) | j = 1, 2, 3, \cdots, r\} = X - \cap \{H_2^m(Q_{n_j}) | j = 1, 2, 3, \cdots, r\}$ so that $L(K) \cap (\cap \{Q_{n_j} | j = 1, 2, 3, \cdots, r\}) \subset L(K) \cap (\cap \{H_2^m(Q_{n_j}) | j = 1, 2, 3, \cdots, r\}) = \phi$. Hence $L(K)$ is τ_1 closed.

Let us define a Kim-type pairwise countable compactness.

DEFINITION 3.7. Let (X, τ_1, τ_2) be a b.t.s. Let $\tau(i, V) = \{\phi, X, \{U \cup V | U \in \tau_i\}\}$ where $V \in \tau_j$ $(i, j = 1, 2;\ i \neq j)$. If $\tau(i, V)$ is countably compact for all $V \in \tau_j (i, j = 1, 2;\ i \neq j)$, then the space is called *pairwise countably compact.*

The proof of the following Proposition is similar to [4, Theorem 2.9].

PROPOSITION 3.8. *In a pairwise countably compact b.t.s.* (X, τ_1, τ_2), *every* τ_i *closed subset is* τ_j *countably compact.*

It can be easily seen that if a b.t.s. is both pairwise E_1 and pairwise countably compact then the two topologies become equal. We state this result without proof.

PROPOSITION 3.9. *If* (X, τ_1, τ_2) *is pairwise* E_1, *then* $\tau_1 = \tau_2$ *if either the space is pairwise countably compact or both the topologies are individually countably compact.*

THEOREM 3.10. *A pairwise* $G\beta$ *b. t. s. is pairwise* G *regular if the space is pairwise countably compact.*

PROOF. Let $x \in X$ and A be a dec. τ_1 closed set with $x \notin A$. Since the space is pairwise $G\beta$, there exists a countable collection $\{W_i | i \in N\}$ of inc. τ_1 open sets each containing x such that $M(x) = \cap \{H_2^m(W_i) | i \in N\}$. Since A is τ_2 countably compact and $X - A \supset M(x)$, there exist positive integers $n_1, n_2, \cdots, n_r$ such that $A \subset \cup \{X - H_2^m(W_{n_j}) | j = 1, 2, 3, \cdots, r\} = U$, say and $x \in \cap \{W_{n_j} | j = 1, 2, 3, \cdots, r\} = V$, say. Then U and V are the required sets given in Definition 2.11. If we had started with an inc. τ_2 closed set B and an element $x \notin B$, in a similar way we would have obtained the required sets U and V of the Definition 2.11. Hence the space is pairwise G regular.

4. Order and quasi-uniformity

In this section we discuss order induced by topological spaces. Pervin [8] and Csaszar [2] proved that every topological space is quasi-uniformizable. The notion of quasi-uniformity was also discussed by Nachbin [6] and he calls the same "semi-uniformity". We call a topological space *weakly orderable* if there exists an order such that the set of all inc. (or dec.) open nbds form a base for the topology. We also call a b. t. s. (X, τ_1, τ_2) *orderable* if and only if the set of all dec. τ_2 open sets form a base for τ_2 and the set of all inc. τ_1 open sets form a base for τ_1.

It will be recalled that a *quasi-uniformity* on a set X is a filter $\mathscr{U}$ on $X\times X$ such that the diagonal $\varDelta$ is contained in each member of $\mathscr{U}$ and for each $U\in\mathscr{U}$, there is a $V\in\mathscr{U}$ such that $V\circ V\subset U$.

A subset C of a quasiordered set X is said to be *convex* if and only if $a, b\in C$ and $a\leqq x\leqq b\Rightarrow x\in C$. The intersection of an inc. and a dec. set is convex. If X is a topological space equipped with a quasiorder, X is said to be *locally convex* [6, p. 26] if and only if the collection of convex nbds of every point of X is a base for the nbd system of this point.

THEOREM 4.1. *Every topological space is weakly orderable. Every pairwise completely regular b. t. s. is orderable.*

PROOF. The proof of the first part of the theorem is actually based on what is given in [6, p. 58]. Let us, for the benefit of the reader, outline the proof. Let (X, τ) be an arbitrary topological space. Then there exists a quasi-uniformity, say, in particular, Pervin's quasi-uniformity, $\mathscr{U}$ which is compatible with τ. Let $G=\bigcap\{U\,|\,U\in\mathscr{U}\}$. Then G is the graph of a quasi-order on X. Let $x\in X$ and A be a τ nbd of x. Then there exists $U\in\mathscr{U}$ such that $U(x)=A$. Then by the definition of quasi-uniformity there exists a $W\in\mathscr{U}$ such that $W\circ W\subset U$. Let us write $B=X-H^{l}(X-W(x))$ where $H^{l}(S)$ stands for the smallest dec. τ closed set that contains S. Then it is easily seen that $B\subset U(x)$. That B is nonempty is also clear. Hence the result.

The proof of the second part is as follows. Let (X, τ_1, τ_2) be pairwise completely regular. Then there exists a quasi-uniformity $\mathscr{U}$ such that $\mathscr{U}$ is compatible with τ_1 and its conjugate $\mathscr{U}^{-1}$ is compatible with τ_2. Let the graph G of the quasiorder be given by $G=\bigcap\{U\,|\,U\in\mathscr{U}\}$. With respect to G, by virtue of (a), the space X is orderable.

THEOREM 4.2. *If (X, τ) is a topological space such that τ is self conjugate in the sense that there exists a quasi-uniformity $\mathscr{U}$ with the property $\mathscr{U}$ and $\mathscr{U}^{-1}$ are both compatible with τ, then X admits an order relation with respect to which the space is locally convex.*

THEOREM 4.3. *The quasiorder relation induced by τ as given in Theorem 4.1 above, is a partial order if and only if the space is T_0.*

PROOF. Let $G=\cap\{U|U\in\mathscr{U}\}$. Since G is a partial order, $G\cap G^{-1}=\Delta$, the diagonal. Hence if $(x,y)\in G$ then $(x, y)\notin G^{-1}$ so that $y\in U(x)$ for some $U\in\mathscr{U}$ and there exists at least one $V\in\mathscr{U}$ such that $x\notin V(y)$.

We now present three results without proof before concluding this section.

PROPOSITION 4.4. *The quasiorder induced by τ as given in Theorem 4.1 is discrete if and only if the space is T_1.*

PROPOSITION 4.5. *The quasiorder induced by τ as given in Theorem 4.1 is an equivalence relation if and only if the space is R_0; and further the equivalence class of x is $cl(\{x\})$.*

PROPOSITION 4.6. *The quasiorder induced by τ as given in Theorem 4.1 is an equivalence relation whose graph is closed in the product space $X\times X$ if and only if the space is R_1.*

5. Pairwise G perfectly normal spaces.

In this section we introduce the notion of pairwise G perfectly normal spaces and study their properties.

DEFINITION 5.1. Let (X,τ_1,τ_2) be a b. t. s. equipped with a quasiorder relation whose graph is G. A set $A(\subset X)$ is called $\tau_i G_\delta^l$ if and only if A is the countable intersection of dec. τ_i open sets. A subset A is called $\tau_i G_\delta^m$ if and only if A is the countable intersection of inc. τ_i open sets.

DEFINITION 5.2. Let (X,τ_1,τ_2) be a b. t. s. equipped with a quasiorder relation whose graph is G. The space is *pairwise G perfectly normal* if and only if it is pairwise normally ordered and every dec. τ_1 closed set is $\tau_2 G_\delta^l$ and every inc. τ_2

closed set is $\tau_1 G_\delta^m$.

DEFINITION 5.3. A b. t. s. (X, τ_1, τ_2) equipped with a quasiorder relation whose graph is G is called *order second countable* if and only if τ_1 has a countable base consisting of inc. τ_1 open sets and τ_2 has a countable base consisting of dec. τ_2 open sets.

The notions of pairwise complete normality and pairwise perfect normality of a b. t. s. not necessarily equipped with a quasiorder were discussed in [5], [7] and [12].

THEOREM 5.4. *If a b. t. s. (X, τ_1, τ_2) equipped with a qusiorder relation (whose graph is G) is pairwise G regular and order second countable, then the space is pairwise G perfectly normal.*

PROOF. Let A be a dec. τ_1 closed set. Let us take $A\neq X$. (If $A=X$, it is trivially $\tau_2 G_\delta^l$). Suppose $x\notin A$. Then there exists a inc. τ_1 open set U_x such that $x\in U_x\subset H_2^m(U_x)\subset X-A$. Let $\alpha=\{V_n \mid n\in N\}$ be a countable base for τ_1 where each V_n is inc. τ_1 open. Then clearly $x\in V_{n(x)}\subset X-A$ for some positive integer $n(x)\in N$ and $V_{n(x)}\in\alpha$. Therefore $H_2^m(V_{n(x)})\subset X-A$. Hence $H_2^m(V_{n(x)})\cap A=\phi$ so that $A=\cap\{X-H_2^m(V_{n(x)}) \mid x\in A\}$. Thus A is $\tau_2 G_\delta^l$. Similarly one can prove that every inc. τ_2 closed subset of X is $\tau_2 G_\delta^m$. Since the space is order second countable, both the topologies are second countable and hence the space is pairwise Lindelöf. Since the space is given to be pairwise G regular, by virtue of Theorem 2.12, the space is pairwise normally ordered and hence the space is pairwise G perfectly normal.

An immediate consequence of the above theorem will be the following.

THEOREM 5.5. *Let (X, τ_1, τ_2) be a pairwise completely regular space. Let τ_1 and τ_2 be second countable. Let $\mathscr{U}$ be the quasi-uniformity compatible with τ_1 such that $\mathscr{U}^{-1}$ be the one compatible with τ_2. Let G be the order generated by $\mathscr{U}$. Then the space is pairwise G perfectly normal.*

THEOREM 5.6. *Let (X, τ_1, τ_2) be a b. t. s. equipped with a quasiorder relation whose graph is G. Then X is pairwise G perfectly normal if and only if for each non empty dec. τ_1 closed set A and a point $b\notin A$, there exists a real-valued* *monotone function f on X such that $f^{-1}(0)=A$, $f(b)=1$, f is τ_2 u. s. c. and τ_1, l. s. c and $f(X)\subset[0,1]$ and for each nonempty inc. τ_2 closed set B and a point $a\notin B$, there is a real-valued antitone function g on X such* that $g^{-1}(0)=B$, *$g(0)=1$, g is τ_1 u. s. c. and τ_2 l. s. c. and $g(X)\subset[0,1]$.*

PROOF. Sufficiency. Let X satisfy the conditions mentioned in the theorem. Let A and B be dec. τ_1 closed and inc. τ_2 closed sets which are disjoint. Let $a_0\in A$ and $b_0\in B$. Then there exists a function $f: X\longrightarrow[0,1]$ such that

(i) f is τ_2 u. s. c. and τ_1 l. s. c.

(ii) f is monotone.

(iii) $f^{-1}(0)=A$ and $f(b_0)=1$.

Also, there exits a function $g: X\longrightarrow[0,1]$ such that

(i) g is τ_1 u. s. c. and τ_2 l. s. c.

(ii) g is antitone.

(iii) $g^{-1}(0)=B$ and $g(a_0)=1$.

Now, if we write $h=f-g$, then clearly h is monotone τ_2 u. s. c. and τ_1 l. s. c. Further $h(a)<0$ for all $a\in A$ and $h(b)>0$ for all $b\in B$. If $h^{-1}(-\infty, 0)=U$ and $h^{-1}(0, \infty)=V$, then U and V are dec. τ_2 open and inc. τ_1 open sets which are disjoint. Further $U\supset A$ and $V\supset B$. Hence the space is pairwise normally ordered.

Let A be any dec. τ_1 closed set. Suppose $b\notin A$. Then there exists a function $f: X\longrightarrow[0,1]$ such that

(i) f is τ_2 u. s. c. and τ_1 l. s. c.

(ii) f is monotnne.

(iii) $f^{-1}(0)=A$ and $f(b)=1$.

Also $A=\bigcap\{f^{-1}(-\infty, \frac{1}{n})\mid n\in N\}$. As f is monotone $f^{-1}(-\infty, \frac{1}{n})$ is dec. for each $n\in N$. Since f is τ_2 u. s. c. $f^{-1}(-\infty, \frac{1}{n})$ is τ_2 open. Thus A is $\tau_2G^l_\delta$. Similarly we can prove that every inc. τ_2 closed set is a $\tau_1G^m_\delta$. Hence the space is pairwise G perfectly normal.

Necessity. Assume the space to be pairwise G perfectly normal. Let A be a nonempty dec. τ_1 closed subset of X. Let $x_0\notin A$ and $A=\bigcap\{V_n\mid n\in N\}$ where each V_n is a dec. τ_2 open set. Thus there exists an integer n_0 such that $x_0\notin V_{n_0}$. Let $G_1=\bigcap\{V_n\mid 1\leq n\leq n_0\}$. G_1 is a dec. τ_2 open set and $A\subset G_1$. Since the space is pairwise normally ordered there exists a dec. τ_2 open set U_1 such that $A\subset U_1\subset$

$H_1^l(U_1)\subset G$. Let us define $G_2=V_{n_0+1}\cap U_1$. Then G_2 is a dec. τ_2 open set such that $A\subset G_2\subset H_1^l(G_2)\subset G_1$. We can now assume that we have defined G_1 to G_k such that G_m is a dec. τ_2 open set containing A and $H_1^l(G_m)\subset G_{m-1}$ and $G_m \subset V_{n_0+m-1}$ for each $m=2, 3, \cdots, k$. Then let $G_{k+1}=V_{n_0+k}\cap U_k$ where U_k is a dec. τ_2 open set such that $A \subset U_k\subset H_1^l(U_k)\subset G_k$. Thus we construct inductively a sequence G_n of dec. τ_2 open sets such that

(i) $A=\cap\{G_n|n\in N\}$

(ii) $H_1^l(G_{n+1})\subset G_n\subset X-M(x_0)$

for each $n\in N$. In the same way as we constructed in the proof of [9, Theorem 3.3], it is possible now to construct a function $f: X\longrightarrow[0,1]$ such that

(i) f is τ_2 u. s. c. and τ_1 l. s. c.

(ii) f is monotone

(iii) $f^{-1}(0)=A$ and $f(x_0)=1$.

Similarly the other parts of the necessary conditions follow.

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