Kyungpook Math. J.
Volume 20 Number 1
June, 1980

A REMARK ON INCOMPARABLE MORPHISMS OF RINGS

By Chul Kon Bae

1. Introduction

Throughout this paper rings will be all commutative rings with units and morphisms will mean unitary ring homomorphisms. The purpose of this paper is to study some properties of an incomparable morphism [5. p. 28, or 1] and a universally incomparable morphism [1]. In this paper, we shall show that if $g: A \longrightarrow C$ is an integral morphism, then $C \longrightarrow B \otimes_A C$ is an incomparable morphism for each $A \longrightarrow B$ in Proposition 3.1 and, in Proposition 3.3, if $f: A \longrightarrow B$ and $g: B \longrightarrow C$ are two universally incomparable morphisms, then $A \longrightarrow B \otimes_A C$ is a unive$_{\text{r}}$sally incomparable morphism. Moreover, we shall discuss an extension of corollary 3.6 [1] in propesition 3.5. It is worth for some general case of B as an A-algebra. Lastly, it will be proved that, for a given morphism $f: A \longrightarrow B$, if $A' \longrightarrow B \otimes_A A'$ is an incomparable morphism (a universally incomparable morphism) for every faithfully flat morphism $g: A \longrightarrow A'$, then f is an incomparable morphism (a universally incomparable morphism) respectively.

I want to express thanks to Professor M. Nishi for his kind advices and constant encouragements, also I am indebted to S. Itoh and A. Ōishi for their stimulating and kind comments.

2. Definitions and Preliminaries

Let A be a commutative ring with identity. We let Spec (A) be the space of all prime ideals of A. For $P \in \operatorname{Spec}(A)$, we denote by $K(P)$ the quotient field of $A/$. Let $f: A \longrightarrow B$ be a morphism. For an ideal J of B, we understand that $J \cap A$ means $f^{-1}(J)$ and we say that J lies over the ideal $J \cap A$ in B and that $J \cap A$ is the contraction of J into A. For a ring A, we denote the Krull dimension of A by dim A.

Let B_i be an A-algebra for each $i \in I$, where I is an ordered set, and (B_i, f_{ji}) be an inductive system with $f_{ji}: B_i \longrightarrow B_j$ A-algebra morphism. We denote the directed limit of B_i's by $B = \varinjlim B_i$.

DEFINITION 2.1. Let $A \longrightarrow B$ be a morphism of rings. We say $A \longrightarrow B$ is an *incomparable morphism* if two different prime ideals of B with the same contraction in A can not be comparable.

DEFINITION 2.2. We say $f: A \longrightarrow B$ is a *universally incomparable morphism* if $C \longrightarrow B \otimes_A C$ is an incomparable morphism for each morphism $g: A \longrightarrow C$.

The following proposition 2.3 is useful [cf. 1].

PROPOSITION 2.3. *Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be two morphisms. Then we have following statements.*

(1) *If f is integral, then f is an incomparable morphism.*

(2) *If f is surjective, then f is an incomparable morphism.*

(3) *If both f and g are incomparable morphisms, then so is gf.*

(4) *If gf is an incomparable morphism, then so is g.*

(5) *If f is a universally incomparable morphism, then f is incomparable.*

The other notations of this paper are similar to [5], [6] and [7].

3. Main results

The following proposition 3.1 and 3.2 are proved easily by proposition 2.3.

PROPOSITION 3.1. *Let $f: A \longrightarrow B$ be an incomparable morphism and $g: A \longrightarrow C$ be an integral morphism. Then $C \longrightarrow B \otimes_A C$ is an incomparable morphism.*

PROOF. Since g is integral, $B \longrightarrow B \otimes_A C$ is integral by change of rings. Then $B \longrightarrow B \otimes_A C$ is an incomparable morphism. Hence, $A \longrightarrow B \otimes_A C$ is an incomparable morphism. Therefore $C \longrightarrow B \otimes_A C$ is an incomparable morphism.

PROPOSITION 3.2. *Let $f: A \longrightarrow B$ be a universally incomparable morphism and $g: B \longrightarrow C$ be an incomparable morphispm. Then $B \longrightarrow B \otimes_A C$ is an incomparable morphism.*

PROOF. $A \longrightarrow C$ and $C \longrightarrow B \otimes_A C$ are incomparable morphisms by proposition 2.3. Thus $B \longrightarrow B \otimes_A C$ is an incomparable morphism.

PROPOSITION 3.3. *If $f: A \longrightarrow B$ and $g: B \longrightarrow C$ are two universally incomparable morphisms, then $A \longrightarrow B \otimes_A C$ is a universally incomparable morphism.*

PROOF. Let $A \longrightarrow D$ be a morphism. Since f and g are universally incomparable morphisms, $D \longrightarrow D \otimes_A C$ is an incomparable morphism. On the other hand, $D \otimes_A C \longrightarrow (D \otimes_A C) \otimes_A B$ is an incomparable morphism from hypothesis. Thus $D \longrightarrow D \otimes_A (C \otimes_A B)$ is an incomparable morphism.

The fo lowing proposition 3.4 is a characterization of an incomparable morphism when an algebra is finitely generated over a field.

PROPOSITION 3.4. *Let A be a finitely generated k-algebra with k a field. Then the following statements are equivalent.*

(1) *$k \longrightarrow A$ is an incomparable morphism.*

(2) *A is integral over k.*

(3) *Spec (A) is finite.*

(4) *Spec (A) is finite and discrete.*

(5) *Spec (A) is discrete.*

(6) *A is Artinian.*

(7) *A is a finite k-algebra.*

(8) *$k \longrightarrow A$ is a universally incomparable morphism.*

PROOF. Obvious. (cf. [1] Theorem 2.9 and [6] p.92)

It follows easily from the definition that $A \longrightarrow B$ is an incomparable morphism if and only if dim $(B \otimes_A K(P))=0$ for each $P \in$ Spec(A). It is useful to prove incomparability in the next propositions.

PROPOSITION 3.5. *Let A be an integral domain. Then the following statements are equivalent.*

(1) *The integral closure of A is a Prüfer domain.*

(2) *For each x contained in the quotient field of A, $A \subseteq A[x]$ is an incomparable extension.*

(3) *For each x contained in the algebraic closure of the quotient field of A, $A \subseteq A[x]$ is an incomparable extension.*

(4) *If B is an extension domain of A and the quotient field of B is algebraic over the quotient field of A, then $A \subseteq B$ is an incomparable extension.*

PROOF. We say that a unitary ring extension $A \subseteq B$ is a P extension if, for each $b \in B$, there exists a polynomial in $A[X]$ having b as a root and one of the coefficents of the polynomial is unit [2]. This definition is equivalent to: For each $b \in B$, there exists a polynomial $f \in A[X]$ having b as a root and $c(f)=A$. Here, $(\Longrightarrow)$ is trivial so we prove $(\Longleftarrow)$. We assume that $f(X)=\sum_{i=0}^{n} a_i X^i$, $c(f)=A$ and $f(x)=0$. Then there exist $b_0, b_1, b_2, \cdots, b_n$ in A such that

$\sum_{i=0}^{n} a_i b_i=1$. We put $g(X)=\sum_{i=0}^{n} b_i X^{n-i} f$; then the coefficient of X^n of $g(X)$ is unit and $g(x)=0$. This last statement is also equivalent to the statement that $A\subseteq A[b]$ is an incomparable extension for each $b\in B$ [1]. By [2] Theorem 5, the integral closure of A in the quotient field k of A is a Prüfer domain iff $A\subset k$ is a P extension and iff $A\subseteq A[x]$ is an incomparable extension for each $x\in k$ by the above arguments. This completes the proof of the equivalence of (1) and (2). Moreover, by [2] theorem 6, the integral closure of A in k is a Prüfer domain iff $A\subset L$ with an algebraic extension L of k is a P extension and iff $A\subset A\ [x]$ is an incomparable extension for each x in L from the above statement. This completes the assertion (1)$\Longleftrightarrow$(3). (3)$\Longrightarrow$(4): If B is a finitely generated A-algebra, then the statement is true by virture of [1] and [4]. General case is obvious by the following lemma.

LEMMA. *Let* $(B_i,\ f_{ji})$ *be a directed system of* A-*algebras and* $B=\varinjlim B_i$. *If* $\dim B_i=0$ *for each* $i\in I$, *then* $\dim B=0$.

PPROOF. For each $P\in\mathrm{Spec}(B)$, let $P_i=P\cap B_i$. Then $B/P=\varinjlim B_i/P_i$ is a field, that is, P is a maximal ideal of B.

Since $A\longrightarrow B_i$ is an incomparable morphism for each $i\in I$, $\dim B_i\otimes_A K(P)=0$ for each $P\in\mathrm{Spec}(A)$. Then $\dim B\otimes_A K(P)=0$ by the above lemma and the fact that $(\varinjlim B_i)\otimes_A K(P)=\varinjlim (B_i\otimes_A K(P))$ [4]. The assertion: (4)$\Longrightarrow$(3) is clear.

The following two propositions are some characterizations of incomparability in terms of flatness.

PROPOSITION 3.6. *Let* A *be an integral domain, and* I *be a non zero ideal of the polynomial ring* $A[X]$. *Then* $A\longrightarrow A[X]/I$ *is an incomparable morphism and* I *is invertible iff* $A\longrightarrow A[X]/I$ *is flat.*

PROOF. The assertion is clear from corollary 2.20 of [3] and corollary 3.2 of [1].

PROPOSITION 3.7. *Let* A *be an integrally closed domain, and* x *be an element in the quotient field of* A. *Then* $A\subseteq A[x]$ *is an incomparable extension iff the inclusion map*: $A\subseteq A[x]$ *is a flat epimorphism.*

PROOF. ($\Longleftarrow$): Trivial.

($\Longrightarrow$): Theorem 67 [5] : Let A be a local integrally closed domain and x be an element of the quotient field of A. Assume that x satisfies a polynomial equation with coefficients in A having at least one coefficient a unit in A. Then either x or x^{-1} lies in A.

Since A is the intersection of localizations of A for each maximal ideal, the domain A is normal iff A_M is normal for every maximal ideal M of A [8, P. 115]. Thus if $x \in \mathrm{A}_M$ then $\mathrm{A}_M = \mathrm{A}_M[x]$ and if $x^{-1} \in \mathrm{A}_M$ then $\mathrm{A}_M[x] = (\mathrm{A}_M)_y$ where $y = x^{-1}$ and $(\mathrm{A}_M)_y$ a localization. Since $A \longrightarrow B$ is flat (an epimorphism, an incomparable morphism) iff $\mathrm{A}_M \longrightarrow B_M$ is flat(an epimorphism, an incomparable morphism) for each maximal ideal M of A respectively, the inclusion map $A \subset A[x]$ is a flat epimorphism.

PROPOSITION 3.8. *Let $f: A \longrightarrow B$ and $g: A \longrightarrow A'$ be two morphisms. If $A' \longrightarrow B \otimes_A A'$ is an incomparable morphism(a universally incomparable morphism) and g is faithfully flat, then f is an incomparable morphism (a universally incomparable morphism) respectively.*

PROOF. (INC). There exists $P' \in \mathrm{Spec}(A')$ for each $P \in \mathrm{Spec}(A)$ by the faithful flatness of g. Since $A' \longrightarrow B \otimes_A A'$ is an incomparable, $\dim(A' \otimes_A B) \otimes_{A'} K(P') = 0$. Moreover $K(P) \otimes_A B \longrightarrow (A' \otimes_A B) \otimes_{A'} K(P')$ is faithfully flat. Thus $\dim K(P) \otimes_A B = 0$, which implies f is incomparable.

(UNIV. INC). Let $A \longrightarrow D$ be a morphism. Since $A' \longrightarrow B \otimes_A A'$ is a universally incomparable morphism, $A' \otimes_A D \longrightarrow (A' \otimes_A D) \otimes_{A'} (B \otimes_A A')$ is an incomparable morphism. On the other hand $D \longrightarrow D \otimes_A A'$ is faithfully flat by the faithful flatness of g. Therefore $D \longrightarrow D \otimes_A B$ is an incomparable morphism by the above statement on (INC).

REFERENCES

[1] Hirohumi Uda, ***Incomparability in Ring Extension.*** Hiroshima Math. J. Vol. 9, No. 2, 1979. pp. 451—463.

[2] Robert Gilmer and Joseph F. Hoffmann, ***A characterization of Prüfer domains in terms of polynomials.*** Pacific J. Math., 60 (1970), pp. 81—85.

[3] Jack Ohm and David E. Rush, ***The finiteness of I when R[X]/I is flat.*** Trans. Amer. Math. Soc., 171 (1972), pp. 377—408.

[4] N. Bourbaki, ***Algebra, Chapter*** 2, Herman, Paris, 1965.

[5] I. Kaplansky, ***Commutative Rings,*** Allyn and Bacon, Boston, 1970.

[6] M. F. Atiyah and I. G. Macdonald, ***Introduction to commutative Algebra,*** Addison-Wesley, 1969.

[7] N. Bourbaki, ***Commutative Algebra, Chapter*** 1, Hermann, Paris, 1965.

[8] Hideyuki Matsumura, ***Commutative Algebra,*** W. A. Benjamin, New York, 1970.