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MULTIPLE GENERALIZED PROLATE SPHEROIDAL WAVE TRANSFORM AND ITS APPLICATION

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0. Abstract

In the present paper the multiple generalized prolate spheroidal wave transform has been developed and its useful operational property has been discussed. As an application of this new transform we have considered the non-homogeneous cubical region. The source of heat generation lies inside it and is dependent upon temperature, and the conductivity is variable.

1. Introduction

Recently, Gupta [5] has defined the generalized Jacobi transform by the equation

$$T\{f(x)\}=\bar{f}_n^{\alpha,\beta}(c)=\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}f(x)\phi_n^{(\alpha,\beta)}(c,x)dx, \tag{1.1}$$

where the generalized prolate spheroidal wave function, $\phi_n^{(\alpha,\beta)}(c, x)$, satisfies the differential equation [5, p. 104]:

$$(1-x^2)\frac{d^2}{dx^2}\phi_n^{(\alpha,\beta)}(c,x)+[(\beta-\alpha)+(\alpha+\beta+2)x]\frac{d}{dx}\phi_n^{(\alpha,\beta)}(c,x)$$
$$+[\chi_n^{(\alpha,\beta)}(c)-c^2x^2)]\phi_n^{(\alpha,\beta)}(c,x)=0, \tag{1.2}$$

$\chi(c)$ being the separation constants for every value of c. $\phi_n^{(\alpha,\beta)}(c,x)$ can be expanded as

$$\phi_n^{(\alpha,\beta)}(c,x)=\sum_{j=0}^{\infty}d_{jn}^{\alpha,\beta}(c)P_{n+j}^{(\alpha,\beta)}(x). \tag{1.3}$$

The coefficients $d_{jn}^{\alpha,\beta}(c)$ can be determined by a five term recursion formula in a manner quite parallel to the case of prolate spheroidal wave functions [4].

The inversion formula for this transform is given by Gupta (himself) as

$$f(x)=\sum_{n=0}^{\infty}\frac{\bar{f}_n^{\alpha,\beta}(c)}{N_n^{(\alpha,\beta)}(c)}\phi_n^{(\alpha,\beta)}(c,x), \tag{1.4}$$

where the normalizing factor, $N_n^{\alpha,\beta}(c)$, is given by

$$N_n^{(\alpha,\beta)}(c)=2^{\alpha+\beta+1}\sum_{j=0}^{\infty}[d_{j,n}^{\alpha,\beta}(c)]^2\frac{\Gamma(n+\alpha+1+j)\,\Gamma(n+j+\beta+1)}{(2n+2j+\alpha+\beta+1)\,\Gamma(n+j+1)}\times\frac{1}{\Gamma(n+j+\alpha+\beta+1)} \qquad (1.5)$$

2. Representation theorem and inversion formula

THEOREM 1. *If $f(x,y,z)$ is continuous (and of bounded variation) over the cubical region $\{(x,y,z): -1<x<1, -1<y<1, -1<z<1\}$, and if the multiple generalized prolate spheroidal wave transform of $f(x,y,z)$ is given by the equation:*

$$T\{f(x,y,z);(x,y,z)\to(p,q,r)\}\equiv\bar{f}(p,q,r)$$
$$=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(1-x)^{\alpha_1}(1-y)^{\alpha_2}(1-z)^{\alpha_3}(1+x)^{\beta_1}(1+y)^{\beta_2}(1+z)^{\beta_3}$$
$$\times f(x,y,z)\,\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)\phi_{n_2}^{(\alpha_2,\beta_2)}(q,y)\phi_{n_3}^{(\alpha_3,\beta_3)}(r,z)\,dx\,dy\,dz, \qquad (2.1)$$

then

$$T^{-1}\{\bar{f}(p,q,r):(p,q,r)\to(x,y,z)\}\equiv f(x,y,z)$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{\bar{f}(p,q,r)}{N_{n_1}^{(\alpha_1,\beta_1)}(p)N_{n_2}^{(\alpha_2,\beta_2)}(q)N_{n_3}^{(\alpha_3,\beta_3)}(r)}$$
$$\times\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)\,\phi_{n_2}^{(\alpha_2,\beta_2)}(q,y)\phi_{n_3}^{(\alpha_3,\beta_3)}(r,z), \qquad (2.2)$$

at the points of the cube at which $f(x,y,z)$ is continuous.

PROOF. By generalized Fourier series [3], the function $f(x,y,z)$ possesses a formal expansion given by

$$f(x,y,z)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}D_{n_1n_2n_3}\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)\phi_{n_2}^{(\alpha_2,\beta_2)}(q,y)\phi_{n_3}^{(\alpha_3,\beta_3)}(r,z),$$
$$(-1\le x\le 1,\ -1\le y\le 1,\ -1\le z\le 1) \qquad (2.3)$$

Now, by grouping the terms in (2.3) so as to display the total coefficients of $\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)$ for each n_1, we can write formally

$$f(x,y,z)=\sum_{n_1=0}^{\infty}\left[\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}D_{n_1n_2n_3}\phi_{n_2}^{(\alpha_2,\beta_2)}(q,y)\phi_{n_3}^{(\alpha_3,\beta_3)}(r,z)\right]\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x). \qquad (2.4)$$

Normalizing the function $\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)$ for all real values of p, we get

$$\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}D_{n_1n_2n_3}\phi_{n_2}^{(\alpha_2,\beta_2)}(q,y)\,\phi_{n_3}^{(\alpha_3,\beta_3)}(r,z)$$
$$=\frac{1}{N_{n_1}^{(\alpha_1,\beta_1)}(p)}\int_{-1}^{1}(1-x)^{\alpha_1}(1+x)^{\beta_1}f(x,y,z)\,\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)dx, \qquad (2.5a)$$
$$=F_{n_1}(y,z),\ \text{say.} \qquad (2.5b)$$

The right hand side of (2.5*a*) is a sequence of functions $F_{n_1}(y, z)$, $n_1=0, 1, \dots,$ each represented by its double Fourier series on the left-hand side of (2.5*a*) on the square $\{(y, z): -1 \leq y \leq 1, -1 \leq z \leq 1\}$, where the coefficients:

$$\sum_{n_3=0}^{\infty} D_{n_1 n_2 n_3} \phi_{n_3}^{\alpha_3, \beta_3}(r, z),$$

of $\phi_{n_2}^{(\alpha_2, \beta_2)}(q, y)$ can be determined in a like manner to give us

$$\sum_{n_3=0}^{\infty} D_{n_1 n_2 n_3} \phi_{n_3}^{(\alpha_3, \beta_3)}(r, z) = \frac{1}{N_{n_1}^{(\alpha_1, \beta_1)}(p) N_{n_2}^{(\alpha_2, \beta_2)}(q)} \int_{-1}^{1} \int_{-1}^{1} (1-x)^{\alpha_1}$$

$$\times (1+x)^{\beta_1}(1-y)^{\alpha_2}(1+y)^{\beta_2} f(x, y, z) \phi_{n_1}^{(\alpha_1, \beta_1)}(p, x) \phi_{n_2}^{(\alpha_2, \beta_2)}(q, y)$$

$$\times \phi_{n_3}^{(\alpha_3, \beta_3)}(r, z) dx dy, \tag{2.6a}$$

$$= F_{n_1 n_2}(z), \text{ say.} \tag{2.6b}$$

Again repeating the above outlined procedure, by eq. (2.1) we finally get

$$D_{n_1 n_2 n_3} = \frac{1}{N_{n_1}^{(\alpha_1, \beta_1)}(p) N_{n_2}^{(\alpha_2, \beta_2)}(q) N_{n_3}^{(\alpha_3, \beta_3)}(r)} \bar{f}(p, q, r) \tag{2.7}$$

Thus, by virtue of eqns. (2.3) and (2.7), the inversion formula (2.2) immediately follows.

3. Particular cases

(i) If $\alpha_1 = \beta_i = 0$ $(i=1, 2, 3)$ and $p=q=r=0$, then the integral transform (2.1) reduces to the well known finite Legendre transform.

(ii) If $p=q=r=0$, then our transform (2.1) reduces to the well-known Jacobi transform.

(iii) If $\alpha_i = \beta_i = -\frac{1}{2} (i=1, 2, 3)$, $p \neq 0$, $q \neq 0$, $r \neq 0$, then our transform (2.1) reduces to the multiple finite Mathieu transform recently defined by me [6].

4. Operational property

THEOREM 2. *If* $f(x, y, z)$ *and its partial derivatives are bounded over the region* $\{(x, y, z): -1<x<1, \quad -1<y<1, \quad -1<z<1\}$, *the generalized prolate spheroidal wave transform of differential operator*:

$$L^{(3)} = (1-x)^{-\alpha_1}(1+x)^{-\beta_1} \frac{\partial}{\partial x}\left[(1-x)^{\alpha_1+1}(1+x)^{\beta_1+1} \frac{\partial f}{\partial x}\right] + (1-y)^{-\alpha_2}(1+y)^{-\beta_2}$$

$$\times \frac{\partial}{\partial y}\left[(1-y)^{\alpha_2+1}(1+y)^{\beta_2+1} \frac{\partial f}{\partial y}\right] + (1-z)^{-\alpha_3}(1+z)^{-\beta_3}$$

$$\times\frac{\partial}{\partial z}\Big[(1-z)^{\alpha_3+1}(1+z)^{\beta_3+1}\frac{\partial f}{dz}\Big]-(p^2x^2+q^2y^2+r^2z^2)f(x,y,z) \quad (4.1)$$

exists and is given by

$$T\{L^{(3)}f(x,y,z)\}=-[\chi_{n_1}^{(\alpha_1,\beta_1)}(p)+\chi_{n_2}^{(\alpha_2,\beta_2)}(q)+\chi_{n_3}^{(\alpha_3,\beta_3)}(f)]\bar{f}(p,q,r), \quad (4.2)$$

provided that

$f(x,y,z)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ *remain finite at both the limits.*

PROOF. By definition (2.1), we first evaluate

$$T\Big\{\frac{1}{(1-x)^{\alpha_1}(1+x)^{\beta_1}}\frac{\partial}{\partial x}\Big[(1-x)^{\alpha_1+1}(1+x)^{\beta_1+1}\frac{\partial f}{\partial x}\Big]-p^2x^2f\Big\}$$

$$=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(1-y)^{\alpha_2}(1+y)^{\beta_2}(1-z)^{\alpha_3}(1+z)^{\beta_3}\frac{\partial}{\partial x}\Big[(1-x)^{\alpha_1+1}(1+x)^{\beta_1+1}\frac{\partial f}{\partial x}\Big]$$

$$\times\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)\,\phi_{n_2}^{(\alpha_2,\beta_2)}(q,y)\phi_{n_3}^{(\alpha_3,\beta_3)}(r,z)dxdydz$$

$$-p^2\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}x^2(1-x)^{\alpha_1}(1+x)^{\beta_1}(1-y)^{\alpha_2}(1+y)^{\beta_2}(1-z)^{\alpha_3}(1+z)^{\beta_3}$$

$$\times f(x,y,z)\,\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)\,\phi_{n_2}^{(\alpha_2,\beta_2)}(q,y)\phi_{n_3}^{(\alpha_3,\beta_3)}(r,z)dxdydz, \quad (4.3)$$

$=I_1+I_2$, say.

Now evaluating the x-integral of I_1 twice by parts, we get

$$I_1=\int_{-1}^{1}\int_{-1}^{1}(1-y)^{\alpha_2}(1+y)^{\beta_2}(1-z)^{\alpha_3}(1+z)^{\beta_3}\phi_{n_2}^{(\alpha_2,\beta_2)}(q,y)\,\phi_{n_3}^{(\alpha_3,\beta_3)}(r,z)$$

$$\times\Big[(1-x)^{\alpha_1+1}(1+x)^{\beta_1+1}\Big\{\frac{\partial f}{\partial x}\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)-f(x,y,z)\frac{\partial}{\partial x}\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)\Big\}\Big]_{-1}^{1}dydz$$

$$+\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\Big[f(x,y,z)\frac{\partial}{\partial x}\Big\{(1+x)^{\beta_1+1}(1-x)^{\alpha_1+1}\frac{\partial}{\partial x}\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)\Big\}$$

$$\times(1-y)^{\alpha_2}(1+y)^{\beta_2}(1-z)^{\alpha_3}(1+z)^{\beta_3}\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)\,\phi_{n_2}^{(\alpha_2,\beta_2)}(q,y)$$

$$\times\phi_{n_3}^{(\alpha_3,\beta_3)}(r,z)\Big]dxdydz \quad (4.4)$$

Since the first expression within curly brackets vanishes on both the limits (because of prescribed conditions), the right-hand side of (4.4) reduces to

$$-\chi_{n_1}^{(\alpha_1,\beta_1)}(p)\bar{f}(p,q,r), \quad (4.5)$$

by virtue of equations (1.2) and (2.1).

Proceeding similarly, we obtain

$$T\left\{(1-y)^{-\alpha_2}(1+y)^{-\beta_2}\frac{\partial}{\partial y}\left[(1-y)^{\alpha_2+1}(1+y)^{\beta_2+1}\frac{\partial f}{\partial y}\right]-q^2y^2\right\}$$
$$=-\chi_{n_2}^{(\alpha_2,\beta_2)}(q)\bar{f}(p,q,r), \tag{4.6}$$

and

$$T\left\{(1-z)^{-\alpha_3}(1+z)^{-\beta_3}\frac{\partial}{\partial z}\left[(1-z)^{\alpha_3+1}(1+z)^{\beta_3+1}\frac{\partial f}{\partial z}\right]-r^2z^2f\right\}$$
$$=-\chi_{n_3}^{(\alpha_3,\beta_3)}(r)\bar{f}(p,q,r). \tag{4.7}$$

Thus, by virtue of results (4.5), (4.6), and (4.7) we arrive at the result (4.2).

5. Application in heat conduction

Here as an application of generalized prolate spheroidal wave transform we consider a non-homogeneous insulated cubical region: $-1\leq x\leq 1$, $-1\leq y\leq 1$, $-1\leq z\leq 1$, where $u(x,y,z,t)$ is the temperature function. The energy equation is given by

$$\frac{\partial u}{dt}=\frac{\partial}{\partial x}(K_x\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(K_y\frac{\partial u}{\partial y})+\frac{\partial}{\partial z}(K_z\frac{\partial u}{\partial z})+Q(x,y,z,t). \tag{5.1}$$

The thermal conductivities K_x, K_y and K_z along the principal axes are proportional to $(1-x^2), (1-y^2)$ and $(1-z^2)$, respectively, *i.e.*, $K_x=K_0(1-x^2)$, $K_y=K_0(1-y^2), K_z=K_0(1-z^2)$.

The source of heat generation, $Q(x,y,z,t)$ depends upon temperature in the form:

$$K_0[(\beta_1-\alpha_1)-(\beta_1+\alpha_1)x]\frac{\partial u}{\partial x}+K_0[(\beta_2-\alpha_2)-(\beta_2+\alpha_2)y]\frac{\partial u}{\partial y}$$
$$+K_0[(\beta_3-\alpha_3)-(\beta_3+\alpha_3)z]\frac{\partial u}{\partial z}-K_0[p^2x^2+q^2y^2+r^2z^2]u$$
$$+K_0\Psi(x,y,z)f(t).$$

The equation (5.1) thus becomes

$$\frac{\partial u}{\partial t}=K_0\Big[(1-x^2)\frac{\partial^2 u}{\partial x^2}+(1-y^2)\frac{\partial^2 u}{\partial y^2}+(1-z^2)\frac{\partial^2 u}{\partial z^2}$$
$$+\{(\beta_1-\alpha_1)-(\beta_1+\alpha_1+2)x\}\frac{\partial u}{\partial x}+\{(\beta_2-\alpha_2)-(\beta_2+\alpha_2+2)y\}\frac{\partial u}{\partial y}$$
$$+\{(\beta_3-\alpha_3)-(\beta_3+\alpha_3+2)z\}\frac{\partial u}{\partial z}-\{p^2x^2+q^2y^2+r^2z^2\}u\Big]$$

$$+K_0\Psi(x,y,z)f(t), \tag{5.2}$$

where $K_0>0$ and is constant. The ends $x=\pm1, y=\pm1$ and $z=\pm1$ are insulated because the conductivity vanishes there.

Now, we solve this equation under the boundary conditions:

(i) $u(x,y,z,t)=0$ at $t=0$, (5.3)

$$\left.\begin{array}{ll}\text{(ii) } K_x\dfrac{\partial u}{\partial x}=0, & \text{at } x=\pm1\\ K_y\dfrac{\partial u}{\partial y}=0, & \text{at } y=\pm1\\ K_z\dfrac{\partial u}{\partial z}=0, & \text{at } z=\pm1\end{array}\right\} \text{ for } t\geq0, \tag{5.4}$$

(iii) $u, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$, and $\dfrac{\partial u}{\partial z}$ remain all finite over the boundaries, *i.e.* at $x=\pm1$, $y=\pm1, z=\pm1$.

Applying the transform (2.1) and its operational property (4.2), the equation (5.2) now becomes

$$\frac{\partial\bar{u}}{\partial t}+K_0[\chi_{n_1}^{(\alpha_1,\beta_1)}(p)+\chi_{n_2}^{(\alpha_2,\beta_2)}(q)+\chi_{n_3}^{(\alpha_3,\beta_3)}(r)]\bar{u}=K_0\bar{\Psi}f(t), \tag{5.5}$$

where $\bar{u}$ and $\bar{\Psi}$ are the generalized prolate spheroidal wave transforms of the functions u and Ψ.

Thus, the appropriate solution of this equation is given by

$$\bar{u}=K_0\bar{\Psi}\int_0^t f(\tau)\exp\left\{-K_0\left[\chi_{n_1}^{(\alpha_1,\beta_1)}(p)+\chi_{n_2}^{(\alpha_2,\beta_2)}(q)+\chi_{n_3}^{(\alpha_3,\beta_3)}(r)\right](t-\tau)\right\}d\tau. \tag{5.6}$$

The inversion formula (2.3) yields

$$u(x,y,z,t)=K_0\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{\bar{u}(p,q,r)}{N_{n_1}^{(\alpha_1,\beta_1)}(p)N_{n_2}^{(\alpha_2,\beta_2)}(q)N_{n_3}^{(\alpha_3,\beta_3)}(r)}$$
$$\times\phi_{n_1}^{(\alpha_1,\beta_1)}(p,x)\phi_{n_2}^{(\alpha_2,\beta_2)}(q,y)\phi_{n_3}^{(\alpha_3,\beta_3)}(r,z)dxdydz, \tag{5.7}$$

where $\bar{u}$ is given by (5.6).

REMARK. The multiple generalized prolate spheroidal wave transform and its operational property defined in secs. 2 and 3, respectively, can be further extended to n-variables.

6. Conclusion

Finally, we conclude that on account of very general nature of kernal in our transform, several other transforms obtained earlier follow as its particular cases. The transform developed in this paper is also applicable to the problems relating to spheroids. The special beauty of this transform is that it eliminates more than one variable at a time. This transform may also be useful to obtain analytical expressions of interest for certain astrophysical situations involving rotating black holes and radiation [1, 2].

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