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# SOME EXPANSION FORMULAE FOR FOX'S $H$-FUNCTION

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## 1. Introduction

In this paper we intend to establish two new expansion formula for Fox's $H$-function. These formulae generalize the results given earlier by Varma [5] and Abiodun and Sharma [1].

The $H$-function introduced by Fox [3, p. 4081], will be represented and defined as follows.

$$(1)\quad H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_1,e_1),\cdots,(a_p,e_p)\\(b_1,f_1),\cdots,(b_q,f_q)\end{matrix}\right]=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=m+1}^{p}\Gamma(a_j-e_js)}z^s ds,$$

where an empty product is interpreted as $1, 0\leq m\leq q, 0\leq n\leq p$; $e$'s and $f$'s are all positive, $L$ is a suitable contour of Barnes type such that the poles of $\Gamma(b_j-f_js)$, $j=1,2,\ldots,m$ lie on the right hand side of the contour and those of $\Gamma(1-a_j+e_js)$, $j=1,2,\ldots, n$ lie on the left hand side of contour.

Recently Braaksma [2] has discussed the asymptotic exapansion and analytical continuation of the $H$-function.

**2.** *The first expansion formula to be proved is*

$$(2)\quad z^\lambda H^{m,n}_{p,q}\left[zx\,\middle|\,\begin{matrix}(a_1,e_1),\cdots,(a_p,e_p)\\(b_1,f_1),\cdots,(b_q,f_q)\end{matrix}\right]$$

$$=h\frac{\prod_{j=1}^{u}\Gamma(\alpha_j)}{\prod_{j=1}^{v}\Gamma(\beta_j)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\ \Gamma(1-\alpha k+h)}{}_{u+2}F_{v+2}\left[\begin{matrix}-k, 1+h(1-\alpha)^{-1},\ \alpha_u;z\\1+h-k\alpha, h(1-\alpha)^{-1}, \beta_v\end{matrix}\right]$$

$$\times H^{m,n+v+2}_{p+v+2,q+u+1}\left[x\,\middle|\,\begin{matrix}(-\lambda,1),(1-h+\alpha k-\lambda,1),(1-\beta_1-\lambda,1),\cdots,(1-\beta_v-\lambda,1),\\(a_1,e_1),\cdots,(a_p,e_p).\\(b_1,f_1),\cdots,(b_q,f_q),(1-\alpha_1-\lambda,1),\cdots,(1-\alpha_u-\lambda,1);\ (k-\lambda,1)\end{matrix}\right]$$

*where* $u=v$ *or* $u+1=v$, $\sum_{j=1}^{p}e_j-\sum_{j=1}^{q}f_j\leq 0$, $h$ *is a positive integer.*

$$\sum_{k=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv s > 0, |\arg x| \leq \frac{1}{2} s\pi,$$

*and the series on the right hand side is convergent.*

PROOF. To prove (2), we substitute the formula due to Abiodun and Sharma [1]

$$(3)\quad z^{\lambda} = h \frac{\prod_{j=1}^{p} \Gamma(a_j) \prod_{j=1}^{q} \Gamma(\beta_j+\lambda)}{\prod_{j=1}^{p} \Gamma(\alpha_j+\lambda) \prod_{j=1}^{q} \Gamma(\beta_j)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(1+\lambda)}{n!\, \Gamma(1+\lambda-n)}$$

$$\times \frac{\Gamma(h-\alpha n+\lambda)}{\Gamma(1-\alpha n+h)}\, {}_{p+2}F_{q+2}\left[\begin{matrix} -n, 1+h(1-\alpha)^{-1}, \alpha_p ; z \\ 1-h-n\alpha, h(1, -\alpha)^{-1}, \beta_q ; \end{matrix}\right]$$

in the left hand side of (2). Expressing the $H$-function as a Mellin-Barnes type integral (1) and interchanging the order of integration and summation, which is justified due to the absolute convergence of the integral and series involved, we have

$$h \frac{\prod_{j=1}^{u} (\alpha_j)}{\prod_{j=1}^{v} (\beta_j)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(1-\alpha k+h)}\, {}_{u+2}F_{v+2}\left[\begin{matrix} -k, 1+h(1-\alpha)^{-1}, \alpha_u ; z \\ 1+h-k\alpha, h(1-\alpha)^{-1}, \beta_v \end{matrix}\right]$$

$$\times \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - f_j s) \prod_{j=1}^{n} \Gamma(1-a_j+e_j s) \prod_{j=1}^{v} \Gamma(\beta_j+\lambda+s) \Gamma(1+\lambda+s) \Gamma(h-\alpha k+\lambda+s) x^s}{\prod_{j=m+1}^{q} \Gamma(1-b_j+f_j s) \prod_{j=n+1}^{p} \Gamma(a_j - e_j s) \prod_{j=1}^{u} \Gamma(\alpha_j+\lambda+s)\, \Gamma(1-k+\lambda+s)} ds.$$

Finally, interpreting the contour integral with the help of (1), we get (2). In particular $p=q=1$ in (2), it reduces to a known result [5, pp. 665 ; eq. (1)].

**3.** *The second expansion formula to be proved is*

$$(4)\quad z^{u}\, H_{p,q}^{m,n}\left[zx \middle| \begin{matrix} (a_1, e_1), \cdots, (a_p, e_p) \\ (b_1, f_1), \cdots, (b_q, f_q) \end{matrix}\right] = \sum_{k=0}^{\infty} (2\lambda+2k) G_{v+2,u}^{t,w+1}\left[z^{-1} \middle| \begin{matrix} 1-\lambda-k, \alpha_v, 1+\lambda+k \\ \beta_u \end{matrix}\right]$$

$$\times H^{m+u-t,\,n+v-w-1}_{p+v+2,\ q+u}\left[x\,\middle|\,\begin{matrix}(1-\lambda-k-\mu,1),(1-\alpha_{w+1}-\mu,1),\cdots,(1-\alpha_v-\mu,1),(a_p,e_p),\\ (1-\alpha_1-\mu,1),\cdots,(1-\alpha_w-\mu,1),(1+\lambda+k-\mu,1),\\ (1-\beta_{t+1}-\mu,1),\cdots,(1-\beta_u-\mu,1);(b_q,f_q).\end{matrix}\right]$$

*valid for* $|z-1|<1, 2(t+w)>u+v, 0\leq(m+u-t)\leq q+u, 0\leq(n+v-w+1)$

$$\leq p+v+2,\ \sum_{j=1}^{p}e_j-\sum_{j=1}^{q}f_q\leq 0,\ \sum_{j=1}^{n}e_j-\sum_{j=n+1}^{p}e_j+\sum_{j=1}^{m}f_j-\sum_{j=m+1}^{q}f_j\equiv s>0,\ |\arg x|\leq\frac{1}{2}s\pi.$$

PROOF. (4) can be proved in the same was as (2) by using the formula due to Sharma [4].

$$(5)\quad x^{\mu}=\frac{\prod_{j=PH}^{S}\Gamma(1-\beta_j-\mu)\prod_{j=q+1}^{I}\Gamma(\alpha_j+\mu)}{\prod_{j=1}^{P}\Gamma(\beta_j+\mu)\prod_{j=1}^{q}\Gamma(1-\mu-\alpha_j)}\sum_{n=0}^{\infty}\frac{(2\lambda+2n)\Gamma(\mu+\lambda+n)}{\Gamma(\lambda-\mu+n+1)}$$

$$\times G^{p,q+1}_{r+2,s}\left[x^{-1}\,\middle|\,\begin{matrix}1-\lambda-n,\alpha_r,1+\lambda+n\\ \beta_s\end{matrix}\right].$$

In particular $p=q=1$, in (4), we get a result due to Abiodun and Sharma [1].

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