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TRANSFORMATIONS OF CERTAIN DOUBLE SERIES.

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1. Introduction

Bailey [2, p. 30, eq. (1.2) (1.4)] have given the following typical formula for ${}_3F_2(1)$ and ${}_4F_3(1)$.

$$(1)\quad {}_3F_2\left[\begin{matrix} a, b, -m\,;\,1 \\ 1+a-b, 1+2b-m\,; \end{matrix}\right] = \frac{(a-2b)_m\,(1+\frac{1}{2}a-b)_m(-b)_m}{(1+a-b)_m(\frac{1}{2}a-b)_m(-2b)_m}$$

and

$$(2)\quad {}_4F_3\left[\begin{matrix} a, 1+\frac{1}{2}a, b, -m\,;\,1 \\ \frac{1}{2}a, 1+a-b, 2+2b-m\,; \end{matrix}\right] = \frac{(a-2b-1)_m\left(\frac{1}{2}+\frac{1}{2}a-b\right)_m(-b-1)_m}{(1+a-b)_m\left(\frac{1}{2}a-b-\frac{1}{2}\right)_m(-2b-1)_m}.$$

The object of this paper is to obtain transformations of certain double series with the help of (1) and (2).

The following notation due to Chaundy [3] is used to represent the hypergeometric series of higher order and of two variables.

$$(3)\quad F\left[\begin{matrix} (a_p)\,;\,(b_q);\ (c_r);\ x, y \\ (d_s);\ (e_h)\,;\,(f_k); \end{matrix}\right] = \sum_{m,n=0}^{\infty} \frac{[(a_p)]_{m+n}[(b_q)]_m[(c_r)]_n x^m y^n}{[(d_s)]_{m+n}[(e_h)]_m[(f_k)]_n m!n!},$$

where (a_p) and $[(a_p)]_{m+n}$ will mean $a_1,\ldots, a_p$ and $(a_1)_{m+n},\ldots,(a_p)_{m+n}$.

2. *The first formula to be proved is*

$$(4)\quad F\left[\begin{matrix} \alpha;\ a-2b, 1+\frac{1}{2}a-b, -b;\ -c-2d, 1+\frac{1}{2}c-d,\ -d;\ 1,\ 1 \\ \beta;\ \frac{1}{2}a-b,\ 1+a-b;\ 1+c-d,\ \frac{1}{2}c-d; \end{matrix}\right]$$

$$=\frac{\Gamma(\beta)\Gamma(\beta-\alpha+2b+2d)}{\Gamma(\beta-\alpha)\Gamma(\beta+2b+2d)}F\left[\begin{matrix} \alpha;\ a,\ b;\ c,\ d;\ 1,1 \\ \beta+2b+2d;\ 1+a-b;\ 1+c-d\,; \end{matrix}\right],$$

valid for $R(\beta-\alpha+2b+2d)>0$.

PROOF. We start with the L.H.S of (4).

$$F\begin{bmatrix}\alpha;\ a-2b,\ 1+\frac{1}{2}a-b,\ -b;\ c-2d,\ 1+\frac{1}{2}c-d,\ -d;\ 1,\ 1\\ \beta;\ 1+a-b,\ \frac{1}{2}a-b;\ 1+c-d,\ \frac{1}{2}c-d;\end{bmatrix}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(a-2b)_n(1+\frac{1}{2}a-b)_n(-b)_n(c-2d)_m(1+\frac{1}{2}c-d)_m(-d)_m}{(\beta)_{m+n}(1+a-b)_n\left(\frac{1}{2}a-b\right)_n(1+c-d)_m\left(\frac{1}{2}c-d\right)_m m!\ n!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\beta)_{m+n}\ m!\ n!}\sum_{r=0}^{n}\frac{(-n)_r(a)_r(b)_r(-2b)_n}{(1+a-b)_r(1+2b-n)_r r!}$$

$$\times\sum_{s=0}^{m}\frac{(-m)_s(c)_s(d)_s(-2d)_s}{(1+c-d)_s(1+2d-m)_s s!}\ \text{by (1)}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha)_{r+s}(a)_r(b)_r(c)_s(d)_s}{(1+a-b)_r(1+c-d)_s(\beta)_{r+s}r!\,s!}$$

$$\times F_1[\alpha+r+s;\ -2b,\ -2d;\ \beta+r+s;\ 1,\ 1].$$

Now we make use of the formula due to Appell Kampe de Feriet [1, p. 22, eq. (24)]

$$(5)\quad F_1[\alpha;\ \beta,\ \gamma;\ \delta;\ 1,\ 1]=\frac{\Gamma(\delta)\Gamma(\delta-\alpha-\beta-\gamma)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta-\gamma)},$$

valid for $R(\delta-\alpha-\beta-\gamma)>0$,

$$=\frac{\Gamma(\beta)\Gamma(\beta-\alpha+2b+2d)}{\Gamma(\beta-\alpha)\Gamma(\beta+2b+2d)}F\begin{bmatrix}\alpha;\ a,\ b:c,\ d:\ 1,1\\ \beta+2b+2d:\ 1+a-b:\ 1+c-d:\end{bmatrix}.$$

This completes the proof of the formula (4). If we put $\alpha=-n$ in (4), it takes the following form.

$$(6)\quad F\begin{bmatrix}-n;\ a-2b,\ 1+\frac{1}{2}a-b,\ -b:c-2d,\ 1+\frac{1}{2}c-d,\ -d:1,1\\ \beta:\ \frac{1}{2}a-b,\ 1+a-b:\frac{1}{2}c-d,\ 1+c-d;\end{bmatrix}$$

$$=\frac{(\beta+2b+2d)_n}{(\beta)_n}F\begin{bmatrix}-n:a,\ b:c,\ d;\ 1,\ 1\\ \beta+2b+2d;\ 1+a-b:1+c-d:\end{bmatrix},$$

valid for $R(\beta+2b+2d+n)>0$.

Next we put $d=0$ in (4), we get following formula of one variable.

$$(7)\ {}_4F_3\left[\begin{matrix}\alpha,\ -b,\ a-2b,\ 1+\frac{1}{2}a-b\,;\,1\\ \beta,\ 1+a-b,\ \frac{1}{2}a-b;\end{matrix}\right]=\frac{\Gamma(\beta)\Gamma(\beta-\alpha+2b)}{\Gamma(\beta-\alpha)\Gamma(\beta+2b)}$$

$$\times{}_3F_2\left[\begin{matrix}\alpha,\ a,\ b;\ 1\\ \beta+2b,\ 1+a-b;\end{matrix}\right],\ \text{valid for } R(\beta-\alpha+2b)>0.$$

In case $\beta=a-2b$ in (7), it reduces to a known result, Bailey [2, p. 45, eq. (2)].

In case $\beta=1+a-\alpha-2b$ in (5) and using Dixons' theorem [2, p. 13, eq. (1)] in (7), it reduces to a known result, Slater [4, p. 245, III. 22].

In case $\alpha=1+a-b$ in (7), it reduces to the following formula.

$$(8)\ {}_3F_2\left[\begin{matrix}\alpha,\ 1+\frac{1}{2}\alpha,\ \gamma\,;\,1\\ \beta,\ \frac{1}{2}\alpha\end{matrix}\right]=\frac{(\beta-\alpha+\gamma-1)\Gamma(\beta)\Gamma(\beta-\alpha-\gamma-1)}{\Gamma(\beta-\gamma)\Gamma(\beta-\alpha)}.$$

In case $\gamma=-n$ in (8), it reduces to a known result, Bailey [2, p. 30, eq. (1.1)].

3. *The second formula to be proved is*

$$(9)\ F\left[\begin{matrix}\alpha\,;\,a-2b-1,\frac{1}{2}+\frac{1}{2}a-b,\ -1-b\,;\,c-2d-1,\frac{1}{2}+\frac{1}{2}c-d,\ -1-d\,;\,1,1\\ \beta\,;\,1+a-b.\ \frac{1}{2}a-b-\frac{1}{2}\,;\,1+c-d,\ \frac{1}{2}c-d-\frac{1}{2}\,;\end{matrix}\right]$$

$$=\frac{\Gamma(\beta)\Gamma(\beta-\alpha+2b+2d+2)}{\Gamma(\beta-\alpha)\Gamma(\beta+2b+2d+2)}$$

$$\times F\left[\begin{matrix}\alpha\,;\ a,\ \frac{1}{2}a+1,b\,;\,c,\frac{1}{2}c+1,d\,;\,1,1\\ \beta+2b+2d+2\,;\,\frac{1}{2}a,\ 1+a-b\,;\,\frac{1}{2}c,\ 1+c-d\,;\end{matrix}\right],$$

valid for $R(\beta-\alpha+2b+2d+2)>0$.

PROOF. (9) can be proved in the same way as (4) on using (2) instead of (1). In particular if $\alpha=-n$ in (9), it takes the following form

$$(10)\ F\left[\begin{matrix}-n\,;\,a-2b-1,\frac{1}{2}+\frac{1}{2}a-b,\ -1-b\,;\,c-2d-1,\frac{1}{2}+\frac{1}{2}c-d,1-d\,;\,1,1\\ \beta\,;\,1+a-b,\ \frac{1}{2}a-b-\frac{1}{2}\,;\,1+c-d,\frac{1}{2}c-d-\frac{1}{2}\,;\end{matrix}\right]$$

$$=\frac{(\beta+2b+2d+2)_n}{(\beta)_n}F\left[\begin{matrix}-n\,;\,a,\ \frac{1}{2}a+1,b\,;\,c,\frac{1}{2}c+1,d\,;\,1,1\\ \beta+2b+2d+2\,;\,\frac{1}{2}a,\ 1+a-b\,;\,\frac{1}{2}c,1+c-d\,;\end{matrix}\right].$$

From (2) we can easily obtain the following transformation

$$(11)\quad {}_4F_3\left[\begin{matrix} a, \frac{1}{2}a+1, b, \alpha\ ;\ 1 \\ \frac{1}{2}a, 1+a-b, \beta+2b+1\ ; \end{matrix}\right]=\frac{\Gamma(\beta+2b+1)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\beta-\alpha+2b+1)}$$

$$\times {}_4F_3\left[\begin{matrix} \alpha, -b-1, a-2b-1, \frac{1}{2}+\frac{1}{2}a-b\ ;\ 1 \\ \beta, 1+a-b, \frac{1}{2}a-b-\frac{1}{2}\ ; \end{matrix}\right].$$

If we put $\beta=a-2b-1$ in (11), we have

$$(12)\quad {}_3F_2\left[\begin{matrix} \frac{1}{2}a+1, b, \alpha\ ;\ 1 \\ \frac{1}{2}a, 1+a-b \end{matrix}\right]=\frac{\Gamma(a)\Gamma(a-\alpha-2b-1)}{\Gamma(a-2b-1)\Gamma(a-\alpha)}$$

$$\times {}_3F_2\left[\begin{matrix} \alpha, -b-1, \frac{1}{2}+\frac{1}{2}a-b\ ;\ 1 \\ 1+a-b, \frac{1}{2}a-b-\frac{1}{2}\ ; \end{matrix}\right].$$

Further we put $\alpha=\frac{1}{2}a-b-\frac{1}{2}$ in (12), we have

$$(13)\quad {}_3F_2\left[\begin{matrix} \frac{1}{2}a+1, b, \frac{1}{2}a-b-\frac{1}{2}\ ;\ 1 \\ \frac{1}{2}a, 1+a-b\ ; \end{matrix}\right]$$

$$=\frac{2^{2b+1}\Gamma\left(\frac{1}{2}a\right)\Gamma(1+a-b)\left(\frac{1}{2}+b+\frac{1}{2}a\right)}{\Gamma\left(\frac{1}{2}a-b\right)\Gamma(2+a)}.$$

Next we put $\alpha=\frac{1}{2}a$ in (12), we have

$$(14)\quad {}_3F_2\left[\begin{matrix} \frac{1}{2}a, 1+b, \frac{1}{2}+\frac{1}{2}a+b\ ;\ 1 \\ 1+a+b, \frac{1}{2}a+b-\frac{1}{2} \end{matrix}\right]$$

$$=\frac{\Gamma(a-2b-1)\Gamma\left(\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma\left(\frac{1}{2}a-2b\right)}{\Gamma(a)\,\Gamma\left(\frac{1}{2}a-2b-1\right)\Gamma\left(\frac{1}{2}a-b\right)\Gamma(a-2b+1)}.$$

In case we put $\alpha=\frac{1}{2}a-b-\frac{1}{2}$ and $\beta=\frac{1}{2}a+\frac{1}{2}-b$ in (11), we get the following formula

$$(15)\quad {}_4F_3\left[\begin{matrix} a, \frac{1}{2}a+1, b, \frac{1}{2}a-b-\frac{1}{2}\ ;\ 1 \\ \frac{1}{2}a, 1+a-b, \frac{3}{2}+\frac{1}{2}a+b\ ; \end{matrix}\right]=\frac{\Gamma(1+a-b)}{\Gamma(2+2b)}$$

$$\times \frac{\Gamma\left(\frac{3}{2}+\frac{1}{2}a+b\right)\Gamma(3+2b)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b\right)\Gamma(2+b)\Gamma(2+a+2b)}.$$

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