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A TWO-DIMENSIONAL NEUMANN EXPANSION FOR THE H-FUNCTION OF TWO VARIABLES

By S. P. Goyal and R. K. Agrawal

1. Introduction

Recently Gupta and Goyal [7], Goyal [5, 6], Srivastava and Daoust [11], Srivastava and Panda [10], Abiodun and Sharma [1] and others have obtained certain Neumann expansions for functions of one or several complex variables. In an attempt to unify some of these results, we first prove a lemma involving a Mellin-Barnes type of double contour integral, popularly known as the H-function of two variables in the literature and make use of this lemma to establish a new two-dimensional generalized Neumann expansion for the said function in a series of products of two H-functions of two variables. This expansion formula is sufficiently general in nature from which one can easily obtain a large number of interesting(known or new) other expansion formulas simply by specializing the parameters of the H-function of two variables.

The parameters of the H-function of two varables [4, p.117] occurring in this paper will be displayed in the following contracted notation:

$$H^{o,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\varepsilon_j)_{1,p_2};(e_j,E_j)_{1,p_3}\\(b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2};(f_j,F_j)_{1,q_3}\end{matrix}\right]$$

$$=H[x,y]=(1/2\pi i)^2\int_{L_1}\int_{L_2}\phi(s,t)\theta_1(s)\theta_2(t)x^sy^t\,ds\,dt \tag{1.1}$$

where

$$\phi(s,t)=\Gamma\left[\begin{matrix}(1-a_j+\alpha_js+A_jt)_{1,n_1}\\(a_j-\alpha_js-A_jt)_{n_1+1,p_1},(1-b_j+\beta_js+B_jt)_{1,q_1}\end{matrix}\right] \tag{1.2}$$

$$\theta_1(s)=\Gamma\left[\begin{matrix}(1-c_j+\varepsilon_js)_{1,n_2},&(d_j-\delta_js)_{1,m_2}\\(c_j-\varepsilon_js)_{n_2+1,p_2},&(1-d_j+\delta_js)_{m_2+1,q_2}\end{matrix}\right] \tag{1.3}$$

$\Gamma\left[\begin{matrix}(a_j)_{n+1,p}\\(b_j)_{m+1,q}\end{matrix}\right]$ stands for $\prod_{j=n+1}^{p}\Gamma(a_j)\left[\prod_{j=m+1}^{q}\Gamma(b_j)\right]^{-1}$

(for integers m,n,p and q such that $0\le n\le p$ and $0\le m\le q$) and $\theta_2(t)$ is defined analogously in terms of parameter sets $(e_j, E_j)_{1,p_3}$ and $(f_j, F_j)_{1,q_3}$.

The symbol $(a_j;\ \alpha_j, A_j)_{n+1,p}$ would abbreviate the $(p-n)$ parameters $(a_{n+1};\ \alpha_{n+1}, A_{n+1}),\ldots,\ (a_p;\ \alpha_p, A_p)$ and $(a_j,\ \alpha_j)_{n+1,p}$, the $(p-n)$ parameters $(a_{n+1}, \alpha_{n+1}),\ldots,\ (a_p, \alpha_p)$ for integers n and p such that $0\leq n\leq p$ and so on.

The conditions on parameters of the H-function of two variables, its asymptotic expansions, some of its properties, particular cases, nature of contours L_1 and L_2 in (1.1) defining the H-function of two variables were given by Goyal [4] in detail. Recently Buschman [2] has explored various sets of condtions sufficient for convergence of a generalized H-function of two variables. This function of Buschman is more general in nature than the function defined by (1.1). The conditions sufficient for the convergence of double Mellin-Barnes integral (1.1), obtained from [2, Eqs. (5.8) or (5.9)] are as follows

$$U=-\sum_{j=n_1+1}^{p_1}\alpha_j-\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{m_2}\delta_j-\sum_{j=m_2+1}^{q_2}\delta_j+\sum_{j=1}^{n_2}\varepsilon_j-\sum_{j=n_2+1}^{p_2}\varepsilon_j>0 \qquad (1.4)$$

$$V=-\sum_{j=n_1+1}^{p_1}A_j-\sum_{j=1}^{q_1}B_j+\sum_{j=1}^{m_3}F_j-\sum_{j=m_3+1}^{q_3}F_j+\sum_{j=1}^{n_3}E_j-\sum_{j=n_3+1}^{p_3}E_j>0 \qquad (1.5)$$

$$|\arg x|<(1/2)U\pi,\ |\arg y|<\frac{1}{2}V\pi. \qquad (1.6)$$

If $n_1=0$ in (1.1) and (1.2), the H-function of two variables will be denoted by $H_1[x, y]$ or $H_1\left[\begin{matrix}x\\y\end{matrix}\right]$ and $\phi(s,t)$ will be denoted by $\phi_1(s,t)$.

Also to save space, three dots "..." appearing at a particular place in any H-function of two variables indicate that the parameters are the same as those of the H-function of two variables defined by (1.1).

2. Results Required

By an appeal to the familiar result due to Reed [9, p. 565] we easily arrive at the following integral

$$\int_0^\infty\int_0^\infty x^{u-1}y^{v-1}H_1[ax^\rho, by^\sigma]\ dx\ dy=\frac{a^{-u/\rho}b^{-v/\sigma}}{\rho\sigma}\phi_1\left(-\frac{u}{\rho},\ -\frac{v}{\sigma}\right)\theta_1\left(-\frac{u}{\rho}\right)\cdot\theta_2\left(-\frac{v}{\sigma}\right) \qquad (2.1)$$

provided that

(i) The conditions (i) and (ii) given on page 119 in the paper by Goyal [4] are satisfied.

(ii) The set of conditions (1.4) to (1.6) are satisfied.

(iii) $\rho, \sigma>0$

$$-\rho \min_{1\le j\le m_2}\left[\mathrm{Re}\left(\frac{d_j}{\delta_j}\right)\right]<\mathrm{Re}(u)<\rho \min_{1\le j\le n_2}\left[\mathrm{Re}\left(\frac{1-c_j}{\varepsilon_j}\right)\right] \text{ and}$$

$$-\sigma \min_{1\le j\le m_3}\left[\mathrm{Re}\left(\frac{f_j}{F_j}\right)\right]<\mathrm{Re}(v)<\sigma \min_{1\le j\le n_3}\left[\mathrm{Re}\left(\frac{1-e_j}{E_j}\right)\right]$$

Also making use of a known result by Gupta and Goyal [7, Eq. (2.3)] we get

$$\int_0^\infty\int_0^\infty x^{\varepsilon-1}y^{\delta-1}J_\lambda(x)J_\mu(y)H_1[ax^\rho,\ by^\sigma]\ dx\ dy=2^{\varepsilon+\delta-2}$$

$$H^{0,0:m_2,n_2+1:m_3,n_3+1}_{p_1,q_1:p_2+2,q_2:p_3+2,q_3}\left[\begin{matrix}2^\rho a\\ 2^\sigma b\end{matrix}\middle|\begin{matrix}\cdots:\left(1-\frac{\lambda+\varepsilon}{2},\ \frac{\rho}{2}\right),\cdots,\left(1+\frac{\lambda-\varepsilon}{2},\ \frac{\rho}{2}\right);\\ \cdots:\qquad\cdots\qquad;\end{matrix}\right.$$

$$\left.\begin{matrix}\left(1-\frac{\mu+\delta}{2},\ \frac{\sigma}{2}\right),\cdots,\left(1+\frac{\mu-\delta}{2},\ \frac{\sigma}{2}\right)\\ \cdots\end{matrix}\right] \tag{2.2}$$

where

(i) The conditions (i) and (ii) given on p. 119 in the paper by Goyal [4] are satisfied.

(ii) The conditions (1.4) to (1.6) given just above are satisfied.

(iii) $\rho,\sigma>0$,

$$-\mathrm{Re}(\lambda)-\rho \min_{1\le j\le m_2}\left[\mathrm{Re}\left(\frac{d_j}{\delta_j}\right)\right]<\mathrm{Re}(\varepsilon)<\rho \min_{1\le j\le n_2}\left[\mathrm{Re}\left(\frac{1-c^j}{\varepsilon_j}\right)\right]+\frac{3}{2}$$

$$-\mathrm{Re}(\mu)-\sigma \min_{1\le j\le m_3}\left[\mathrm{Re}\left(\frac{f_j}{F_j}\right)\right]<\mathrm{Re}(\delta)<\sigma \min_{1\le j\le n_3}\left[\mathrm{Re}\left(\frac{1-e_j}{Ej}\right)\right]+\frac{3}{2}$$

Again, by an appeal to the Neumann expansion given by Abiodun and Sharma [1, p. 260], we have

$$x^u y^v=2^{u+v}\sum_{m,n=0}^\infty(\lambda+2m)(\mu+2n)\Gamma\left[\begin{matrix}\frac{\lambda+u}{2}+m,\ \frac{\mu+v}{2}+n\\ 1+\frac{\lambda-u}{2}+m,\ 1+\frac{\mu-v}{2}+n\end{matrix}\right]J_{\lambda+2m}(x)\cdot J_{\mu+2n}(y) \tag{2.3}$$

Now, we give generalization of the above Neumann expansion in terms of the H-function of two variables:

LEMMA. *If* $\rho,\ \sigma>0$,

$$\rho \min_{1\le j\le m_2}\left[\mathrm{Re}\left(\frac{d_j}{\delta_j}\right)\right]>\max\,[\mathrm{Re}(-u,\ -\lambda)],$$

$$\sigma \min_{1\le j\le m_3}\left[\mathrm{Re}\left(\frac{f_j}{F_j}\right)\right]>\max\,[\mathrm{Re}(-v,\ -\mu)],$$

$$\rho \min_{1\leq j\leq n_2}\left[\mathrm{Re}\left(\frac{1-c_j}{\varepsilon_j}\right)\right] > \max\left[\mathrm{Re}\left(u, -\frac{3}{4}\right)\right] \textit{ and}$$

$$\sigma \min_{1\leq j\leq n_3}\left[\mathrm{Re}\left(\frac{1-e_j}{E_j}\right)\right] > \max\left[\mathrm{Re}\left(v, -\frac{3}{4}\right)\right]$$

then

$$x^{\frac{u}{\rho}} y^{\frac{v}{\sigma}} = 4\rho\sigma\left[\phi_1\left(-\frac{u}{\rho}, -\frac{v}{\sigma}\right)\theta_1\left(-\frac{u}{\rho}\right)\theta_2\left(-\frac{v}{\sigma}\right)\right]^{-1} \sum_{m,n=0}^{\infty}(\lambda+m)(\mu+n)$$

$$\Gamma\begin{bmatrix}\lambda+u+m, & \mu+v+n\\ \lambda-u+m+1, & \mu-v+n+1\end{bmatrix} H^{0,0:m_2,n_2+1:m_3,n_3+1}_{p_1,q_1:p_2+2,q_2:p_3+2,q_3}\left[\begin{matrix}x^{-1}\\ y^{-1}\end{matrix}\middle|\begin{matrix}\cdots:(1-\lambda-m,\rho),\\ \cdots:\end{matrix}\right.$$

$$\left.\begin{matrix}\ldots,(1+\lambda+m,\rho):(1-\mu-n,\sigma),\ldots,(1+\mu+n,\sigma)\\ \ldots \qquad : \qquad \ldots\end{matrix}\right] \tag{2.4}$$

provided that

(i) *the conditions* (i) *and* (ii) *given on p.* 119 *in the paper by Goyal* [4] *and the conditions* (1.4) *through* (1.6) *are satisfied.*

(ii) *The double series on the right-hand side of* (2.4) *is absolutely convergent.*

PROOF. To prove (2.4), replace x, y, u, v, λ and μ by ξ, η, $2u$ $2v$, 2λ and 2μ respectively, multiply both the sides of (2.3) by $\xi^{-1}\eta^{-1}H_1\left[\frac{\xi^{2\rho}}{x}, \frac{\eta^{2\sigma}}{y}\right]$ and integrate with respect to ξ and η over 0 to ∞, then use the formula (2.1) and (2.2), we get the desired result (2.4).

If we set $p_1=q_1=0$ in (2.4), use a known relult [4, p.123, Eq.(3.4)] in it, the double series breaks into two independent series over m and n, and therefore we have

$$x^{\frac{u}{\rho}} = 2\rho\sum_{m=0}^{\infty}(\lambda+m)\Gamma\begin{bmatrix}\left(1-d_j-\delta_j\frac{u}{\rho}\right)_{m_2+1,q_2}, & \left(c_j+\varepsilon_j\frac{u}{\rho}\right)_{n_2+1,p_2}, & u+\lambda+m\\ \left(d_j+\delta_j\frac{u}{\rho}\right)_{1,m_2}, & \left(1-c_j-\varepsilon_j\frac{u}{\rho}\right)_{1,n_2}, & 1+\lambda-u+m\end{bmatrix}$$

$$\cdot H^{m_2,n_2+1}_{p_2+2,q_2}\left[x^{-1}\middle|\begin{matrix}(1-\lambda-m,\rho), & (c_j,\varepsilon_j)_{1,p_2}, & (1+\lambda+m,\rho)\\ & (d_j,\delta_j)_{1,q_2} & \end{matrix}\right] \tag{2.5}$$

where,

$$A=\sum_{1}^{m_2}\delta_j-\sum_{m_2+1}^{q_2}\delta_j+\sum_{1}^{n_2}\varepsilon_j-\sum_{n_2+1}^{p_2}\varepsilon_j>0,$$

$$|\arg x|<\frac{1}{2}A\pi,\ \rho>0,\ \rho\min_{1\leq j\leq m_2}\left[\mathrm{Re}\left(\frac{d_j}{\delta_j}\right)\right]>\max\left[\mathrm{Re}(-u, -\lambda)\right]$$

and

$$\rho \min_{1\leq j\leq n_2}\left[\mathrm{Re}\left(\frac{1-c_j}{\varepsilon_j}\right)\right]> \max\left[\mathrm{Re}\left(u,\ -\frac{3}{4}\right)\right].$$

Further, if we put all ε's, δ's and ρ equal to one in (2.5) we arrive at a known result by Abiodun and Sharma [1, p. 260].

Also, if $\rho=1$, $u=\lambda$ in (2.5), we arrive at the Neumann expansion in terms of H-function given earlier by Srivastava and Daoust [11, p. 453, Eq. (2.7)].

3. The Main Expansion Formula

$$x^u y^v H^{o\ ,\ n_1+m'_2+m'_3\ :\cdots;\cdots}_{p_1+q'_1+q'_2+q'_3,\ q_1+p'_1+p'_2+p'_3:\cdots;\cdots}\left[\begin{matrix} a x^{\varepsilon} y^{\nu} \\ b x^{\delta} y^{\tau}\end{matrix}\middle|\begin{matrix} L:\cdots;\cdots \\ M:\cdots;\cdots\end{matrix}\right]$$

$$=4\sum_{m,n=0}^{\infty}(\lambda+m)(\mu+n)\ H^{o,o:m'_2,\ 1:m'_3,1}_{p'_1,q'_1:p'_2+2,q'_2:p'_3+2,q'_3}\left[\begin{matrix} x^{-1} \\ y^{-1}\end{matrix}\middle|\begin{matrix} (1-a'_j-u\alpha'_j-vA'_j; \\ (1-b'_j-u\beta'_j-vB'_j;\end{matrix}\right.$$

$$\begin{matrix} \alpha'_j, A'_j)_{1,p'_1}: (1-\lambda-m,1),\ (1-c'_j-u\varepsilon'_j, \varepsilon'_j)_{1,p'_2},\ (1+\lambda+m,1);\ (1-\mu-n,1), \\ \beta'_j, B'_j)_{1,q'_1}; \qquad (1-d'_j-u\delta'_j, \delta'_j)_{1,q'_2} \qquad ; \end{matrix}$$

$$\left.\begin{matrix} (1-e'_j-vE'_j, E'_j)_{1,p'_3},\ (1+\mu+n,1) \\ (1-f'_j-vF'_j, F'_j)_{1,q'_3} \end{matrix}\right] H^{o,n_1+2:\cdots;\cdots}_{p_1+4,q_1:\cdots;\cdots}\left[\begin{matrix} a \\ b\end{matrix}\middle|\begin{matrix} F_{m,n}:\cdots;\cdots \\ \cdots\ :\cdots;\cdots\end{matrix}\right]. \tag{3.1}$$

where L stands for $(a_j; \alpha_j, A_j)_{1,n_1}$, $(d'_j; \varepsilon\delta'_j, \delta\delta'_j)_{1,m'_2}$, $(f'_j; \nu F'_j, \tau F'_j)_{1,q'_3}$, $(d'_j; \varepsilon\delta'_j, \delta\delta'_j)_{m'_2+1,q'_2}$, $(a_j; \alpha_j, A_j)_{n_1+1,p_1}$, $(b'_j; \varepsilon\beta'_j+\nu B'_j,\ \delta\beta'_j+\tau B'_j)_{1,q'_1}$;

M for $(b_j;\beta_j, B_j)_{1,q_1}$, $(a'_j; \varepsilon\alpha'_j+\nu A'_j,\ \delta\alpha'_j+\tau A'_j)_{1,p'_1}$, $(c'_j; \varepsilon\varepsilon'_j, \delta\varepsilon'_j)_{1,p'_2}$, $(e'_j; \nu E'_j, \tau E'_j)_{1,p'_3}$ *and*

$F_{m,n}$ for $(1-\lambda-m-u; \varepsilon, \delta)$, $(1-\mu-v-n; \nu, \tau)$, $(a_j; \alpha_j, A_j)_{1,p_1}$, $(1+\lambda-u+m; \varepsilon, \delta)$, $(1+\mu-v+n; \nu, \tau)$.

The formula (3.1) *holds for all finite values of x and y, if*

(i) *The conditions* (i) *and* (ii), *modified appropriately, given on p.* 119 *in the paper by Goyal* [4] *are satisfied by H-function of two variables occurring on the left-hand side of* (3.1).

(ii) *The conditions* (1.4) *to* (1.6), *modified appropriately are satisfied by the H-function of two variables occurring on the left-hand side of* (3.1).

(iii) $\mathrm{Re}(u)\geq 0$, $\mathrm{Re}(v)\geq 0$, $\varepsilon, \delta, \nu, \tau\geq 0$ (*not all zero simultaneously*)

(iv) *The double series on the right-hand side of* (3.1) *is absolutely convergent.*

PROOF. Writing contour integral for $H\left[\begin{matrix} a \\ b\end{matrix}\right]$ involved in the right-hand side of

(3.1) by (1.1), changing the order of integration and summation, which is permissible under the conditions mentioned with (3.1), we get

$$r.h.s. \text{ of } (3.1)=(1/2\pi i)^2\int_{L_1}\int_{L_2}\phi(s,t)\theta_1(s)\ \theta_2(t)a^s b^t$$

$$\left\{4\sum_{m,n=0}^{\infty}(\lambda+m)(\mu+n)\Gamma\begin{bmatrix}\lambda+m+u+\varepsilon s+\delta t, & \mu+n+v+\nu s+\tau t\\ 1+\lambda+m-u-\varepsilon s-\delta t, & 1+\mu+n-v-\nu s-\tau t\end{bmatrix}\right.$$

$$\left.\cdot H^{o,o\ :\ m'_2,1\ ;\ m'_3,1}_{p'_1,q'_1\ :\ p'_2+2,q'_2\ ;\ p'_3+2,q'_3}\begin{bmatrix}x^{-1}\\ y^{-1}\end{bmatrix}\right\} ds\ dt \tag{3.2}$$

where the remaining parameters of $H\begin{bmatrix}x^{-1}\\ y^{-1}\end{bmatrix}$ are the same as those given in (3.1).

Now applying the lemma (2.4) with $\rho=\sigma=1$, $n_2=n_3=0$, we easily arrive at the left-hand side of (3.1).

REMARK. The expansion formula (3.1) can also be generalized to yield the multi-dimensional expansional formula for the H-function of several complex variables [10]. The formula can be developed by following the lines similar to those indicated while proving the formula (3.1). We however omit the details, on account of triviality of analysis involved.

4. Some Interesting Deductions.

In the expansion formula (3.1), if we set $p'_1=q'_1=\nu=\tau=v=0$, apply a known result [4, p.123, Eq. (3.4)] and sum the series over n with the help of a special case ($\rho=1$, $n_2=0$) of (2.5), we are led to the following resu t:

$$x^u H^{o,n_1+m'_2\ :\ \cdots\ ;\ \cdots}_{p_1+q'_2,q_1+p'_2\ :\ \cdots\ ;\ \cdots}\left[\begin{matrix}ax^{\varepsilon}\\ bx^{\delta}\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,n_1},\ (d'_j;\varepsilon\delta'_j,\delta\delta'_j)_{1,q'_2},\ (a_j;\alpha_j,A_j)_{n_1+1,p_1}:\cdots;\cdots\\ \cdots,(c'_j;\varepsilon\varepsilon'_j,\delta\varepsilon'_j)_{1,p'_2}\qquad :\cdots;\cdots\end{matrix}\right]=2\sum_{m=0}^{\infty}(\lambda+m)\ H^{1,\ m'_2}_{q'_2,p'_2+2}\left[x\middle|\begin{matrix}(d'_j+u\delta'_j,\delta'_j)_{1,q'_2}\\ (\lambda+m,1),\ (c'_j+u\varepsilon'_j,\varepsilon'_j)_{1,p'_2},\ (-\lambda-m,1)\end{matrix}\right]H^{o,n_1+1\ :\ \cdots\ ;\ \cdots}_{p_1+2,q_1\ :\ \cdots\ ;\ \cdots}\left[\begin{matrix}a\\ b\end{matrix}\middle|\begin{matrix}(1-\lambda-m-u;\varepsilon,\delta),\cdots,(1+\lambda+m-u;\varepsilon,\delta):\cdots;\cdots\\ \cdots\qquad :\cdots;\cdots\end{matrix}\right] \tag{4.1}$$

provided that

(i) *The conditions* (i) *and* (ii), *modified appropriately given on p.* 119 *in the* *paper by Goyal* [4] *are satisfied by the H-function of two variables occurring on the left-hand side of* (4.1).

(ii) *The conditions* (1.4) *to* (1.6), *modified appropriately are satisfied by the same H-function of two variables.*

(iii) $\mathrm{Re}(u)\geq 0$, $\varepsilon, \delta \geq 0$ (*not both zero simultaneously*).

(iv) $A' = \sum_{j=1}^{m'_2} \delta'_j - \sum_{j=m'_2+1}^{q'_2} \delta'_j - \sum_{j=1}^{p'_2} \varepsilon'_j > 0$, $|\arg x| < \frac{1}{2} A'\pi$.

(v) *The series on the right-hand side of* (4.1) *is absolutely convergent.*

If in (4.1), we put $\varepsilon=\delta=m_2=m_3=1$, $n_j=p_j$ $(j=1,2,3)$, $\delta'_i=\varepsilon'_j=1$ $(i=1,\cdots,q'_2;$ $j=1,\cdots,p'_2)$, $\lambda=u$, $d_1=f_1=0$, $\delta_1=F_1=1$, and make use of results by Goyal [4, p.120, Eq.(1.2); p.123, Eq.(3.3)], we get a result due to Srivastava and Daoust [1, p.453, Eq.(3.1)] after a little simplification and change in parameters. This result itself is generalization of many other known expansion formulas.

Further, if we put $m'_2=p'_2=q'_2=0$, replace x by $x^2/4$ in the expansion formula (4.1), use the results [8, p.598, Eq.(4.1); 3, p.219, Eq.(44)] therein and make slight change in parameters, we easily arrive at the following result:

$$x^u H[ax^{\varepsilon}, bx^{\delta}] = 2^u \sum_{m=0}^{\infty} (\lambda+2m) J_{\lambda+2m}(x) H^{o,n_1+1:\cdots;\cdots}_{p_1+2,q_1:\cdots;\cdots}\left[\begin{matrix}2^{\varepsilon}a\\ 2^{\delta}b\end{matrix}\right| \left(1-\frac{\lambda+u}{2}-m;\ \frac{\varepsilon}{2},\ \frac{\delta}{2}\right),\cdots,\left(1+\frac{\lambda-u}{2}+m;\ \frac{\varepsilon}{2},\ \frac{\delta}{2}\right);\cdots;\cdots \atop \cdots \qquad :\cdots;\cdots\Bigg] \tag{4.2}$$

The formula (4.2) holds for all finite values x and y for the conditions easily deducable from (4.1).

It is quite obvious that the special case of formula (4.2) will yield a special case of formula ($r=2$) due to Srivastava and Panda [10, p.176, Eq.(3.16)]. Again the formula (4.2) gives a result due to Goyal [5, Eq. (3.1)] when $p_1=q_1=0$. Evidently (4.2) also includes the expansion formula by Abiodun and Sharma [1, p.260] and others.

On the other hand, if we put $p'_1=q'_1=0$ in (4.1), we get the following formula after applying the result [8, p.595, Eq.(2.2)] in it:

$$x^u y^v H^{o,n_1+m'_2+m'_3:\cdots;\cdots}_{p_1+q'_2+q'_3,q_1+p'_2+p'_3:\cdots;\cdots}\left[\begin{matrix}ax^{\varepsilon}y^{\nu}\\ bx^{\delta}y^{\tau}\end{matrix}\right|\left.\begin{matrix}L':\cdots;\cdots\\ M':\cdots;\cdots\end{matrix}\right]$$

$$=4\sum_{m,n=o}^{\infty}(\lambda+m)(\mu+n) H^{1,m'_2}_{q'_2,p'_2+2}\left[x\ \middle|\ \begin{matrix}(d'_j+u\delta'_j,\delta'_j)_{1,q'_2}\\ (\lambda+m,1),\ (c'_j+u\varepsilon'_j,\varepsilon'_j)_{1,p'_2},(-\lambda-m,1)\end{matrix}\right]$$

$$H^{1, m'_3}_{q'_3, p'_3+2}\left[y \,\middle|\, \begin{matrix} (f_j+vF'_j, F'_j)_{1, q'_3} \\ (\mu+n, 1),\ (e'_j+vE'_j,\ E'_j)_{1, p'_3},\ (-\mu-n, 1) \end{matrix} \right]$$

$$H^{o, n_1+2 : \cdots ; \cdots}_{p_1+4, q_1 : \cdots ; \cdots}\left[\begin{matrix} a \\ b \end{matrix} \,\middle|\, \begin{matrix} F_{m, n} : \cdots ; \cdots \\ \cdots : \cdots ; \cdots \end{matrix} \right] \tag{4.3}$$

where L' stands for $(a_j; \alpha_j, A_j)_{1, n_1}$, $(d'_j; \varepsilon\delta'_j, \delta\delta'_j)_{1, m'_2}$, $(f'_j; \nu F'_j, \tau F'_j)_{1, q'_3}$, $(d'_j; \varepsilon\delta'_j, \delta\delta'_j)_{m'_2+1, q'_2}$, $(a_j; \alpha_j, A_j)_{n_1+1, p_1}$. M' for $(b_j; \beta_j, B_j)_{1, q_1}$, $(c'_j; \varepsilon\varepsilon'_j, \delta\varepsilon'_j)_{1, p'_2}$, $(e'_j; \nu E'_j, \tau E'_j)_{1, p'_3}$ and $F_{m, n}$ for the quantities mentioned with (3.1).

The conditions of validity of (4.3) can be easily obtained from those given with (3.1).

The expansion formula (4.3) is also sufficiently general in nature and would reduce to recently obtained formula of Gupta and Goyal [7], if we appeal to the known results [8, p. 598, Eq. (4.1); 3, p. 219, Eq. (44)]. This special case also includes a result of Goyal [6, p. 122, Eq. (3.1)].

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Department of Mathematics
College of Arts & Science
Banasthali Vidyapith-304022
Rajasthan, India.

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