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A FIXED POINT THEOREM IN METRICALLY CONVEX METRIC SPACES

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0. Introduction

In a recent paper Maiti et al [4] proved the following theorm.

THEOREM A. *Let* (x,ρ) *be a complete metric space. Given an* $\varepsilon>0$ *there exists a* $\delta>0$ *such that the mapping* $f: X\to X$ *satisfies the condition*

(1) $$\varepsilon\leq\max\{\rho(x,y),\rho(x,fx),\rho(y,fy)\}<\varepsilon+\delta\Longrightarrow\rho(fx,fy)<\varepsilon$$

for $x,y\in X$. *Also let the functional* $F(x)=\rho(x,fx)$ *is lower semi-continuous. Then* f *has a unique fixed point.*

It is worthwhile to remark that theorem A is more general than weakly uniformly strict contraction of Meir and Keeler [3] (which is valid only for continuous mapping). We give an example to illustrate this fact.

EXAMPLE. Let $X=[0,7]$ and the function f is defined by

$$f(x)=\begin{cases}\frac{2}{5}x, & \text{if } 0\leq x\leq 6\\ -2x+14 & \text{if } 6<x\leq 7\end{cases}$$

In this case the theorem of Meir and Keeler [3] is not satisfied since f is discontinuous at $x=6$, but theorem A is satisfied.

The chief aim of the present note is to prove a new fixed point theorem for mappings satisfying (1) in metrically convex metrix spaces. This result further generalizes the result of Assad [1].

1. Fixed point theorem

THEOREM 1. *Let* (X,ρ) *be a complete, metrically convex metric space and* M *a non-empty closed subset of* X. *Let* $f: M\to M$ *satisfies* (1) *and* $fx\in M$ *for every* $x\in\delta M$ *(*δM *denotes the boundary of* M*). Also let* (ii) *if* $x\in\delta M$, $y\in M$, $z\in M$, $fz\notin M$ *then* $\rho(x,y)<\rho(fz,y)$. *Further if the functional* $F(x)=\rho(x,fx)$ *is lower semi-continuous, then* f *has a unique fixed point.*

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PROOF. We construct a sequence $\{x_n\}$ in M as follows: let x_0 be an arbitrary point in M. Let $x'_1=f(x_0)$. If $x_1'\in M$, then set $x_1=x_1'$, otherwise we choose $x_1\in\delta M$ so that $\rho(x_0,x_1)+\rho(x_1,x_1')=\rho(x_0,x'_1)$ (cf. [2]). Suppose that $\{x_i\}$, $\{x'_i\}$, $i=1,2,\cdots,N$ have been so chosen that

(a) $$x_i'=f(x_{i-1}), i=1,2,\cdots,N$$

(b) either $x_i=x'_i\in M$ or $x_i\in\delta M$ and satisfies the relation

$$\rho(x_{i-1},x_i)+\rho(x_i,x'_i)=\rho(x_{i-1},x'_i).$$

Now let $x'_{N+1}=f(x_N)$. If $x_{N+1}\in M$ we put $x_{N+1}=x'_{N+1}$, otherwise we choose $x_{N+1}\in\delta M$ so that

$$\rho(x_N,x'_{N+1})=\rho(x_N,x_{N+1})+\rho(x_{N+1},x'_{N+1}).$$

and hence by induction we are finished.

If there exists $x_j\in\{x_n\}$ such that all of its iterates lie in M, Maiti et al [4] showed that this sequence of iterates converges to a fixed point of f. Therefore we may assume that there exists infinitely many points $x_i\in\{x_n\}$ for which $x_i=x'_i$. Let $\{x_{n_k}\}$ be the subsequence of all such points in $\{x_n\}$, i.e. $x_{n_k}\neq x'_{n_k}$. We assume that

(2) $$\rho(x_n,x_{n+1})\to 0 \text{ as } n\to\infty$$

and

(3) $$\rho(fx_n,x_n)\to 0 \text{ as } n\to\infty$$

To establish (2) and (3) we first show that

(4) $$\rho(x_{n_k-1},x'_{n_k})\to 0 \text{ as } k\to\infty$$

To prove (4) we use the fact that the inequality (1) implies

$$\rho(fx,fy)<\max\{\rho(x,y),\rho(x,fx),\rho(y,fy)\}.$$

If we put $n_k=p$, $n_k+1=q$ then it follows from (4) that

(5) $$\begin{aligned}\rho(x_{q-1},x'_q)&=\rho(f(x_{q-2}),f(x_{q-1}))\\&<\max\{\rho(x_{q-2},x_{q-1}),\rho(x_{q-2},x_{q-1}),\rho(x_{q-1},x'_q)\}\\&<\rho(x_{q-2},x_{q-1})<\cdots\cdots<\rho(x_p,x_{p+1}).\end{aligned}$$

Now by condition (ii) we have if $x_p\neq x'_p$, then $\rho(x_p,x_{p+1})<\rho(x'_p,x_{p+1})$. Also we have for $x_p\neq x'_p$, $\rho(x_p,x_{p+1})<\rho(x_{p-1},x'_p)$. Because we have

(6) $$\begin{aligned}\rho(x_p,x_{p+1})<\rho(x_p',x_{p+1})&=\rho(f(x_{p-1}),f(x_p))\\&<\max\{\rho(x_{p-1},x_p),\rho(x_{p-1},x'_p),\rho(x_p,x_{p+1})\}\end{aligned}$$

$$<\rho(x_{p-1}, x_p)+\rho(x_p, x'_p)$$
$$=\rho(x_{p-1}, x'_p)$$

Putting the value of (6) in (5) we get

$$\rho(x_{q-1}, x_q)<\rho(x_{p-1}, x'_p).$$

Hence the sequence $\{\rho(x_{n_k-1}, x'_{n_k})\}$ is decreasing. Suppose that $\rho(x_{n_k-1}, x'_{n_k})\to \varepsilon>0$ Then for all $k=1, 2, \cdots\cdots$

$$\rho(x_{n_k-1}, x'_{n_k})=\max\{\rho(x_{n_k-1}, x'_{n_k}), \rho(x_{n_k-1}, f(x_{n_k-1})), \rho(x'_{n_k}, f(x'_{n_k}))\}>\varepsilon.$$

But the condition (1) implies that

$$\varepsilon\leq\max\{\rho(x, y), \rho(x, fx), \rho(y, fy)\}<\varepsilon+\delta\Longrightarrow\rho(f(x), f(y))<\varepsilon.$$

We know that there exists an integer N such that for $k\geq N$

$$\rho(x_{n_k-1}, x'_{n_k})=\max\{\rho(x_{n_k-1}, x'_{n_k}), \rho(x_{n_k-1}, f(x_{n_k-1})), \rho(x'_{n_k}, f(x'_{n_k}))\}<\varepsilon+\delta.$$

So if we let $n_k=p, n_k+1=q$ it follows from (6) that

$$\rho(x_p, x_{p+1})<\rho(x_{p-1}, x'_p)<\varepsilon+\delta.$$

Also on the other hand

$$\varepsilon<\rho(x_{q-1}, x'_q)<\rho(x_p, x_{p+1}).$$

Therefore

$$\varepsilon<\rho(x_p, x_{p+1})<\varepsilon+\delta.$$

It follows that

$$\rho(x_{q-1}, x'_q)<\cdots\cdots<\rho(x_{p+1}, x'_{p+2})=\rho(f(x_p), f(x_{p+1}))<\varepsilon$$

and this contradicts the assumption that for all $k=1, 2, \cdots\cdots, \rho(x_{n_k-1}, x'_{n_k})>\varepsilon$. Hence we have proved (4). Now we shall show that (2) follows from (4).

Let (2) is not true, then there exists an $\varepsilon>0$ such that for every positive integer N, there exists $n\geq N$ such that $\rho(x_n, x_{n+1})>\varepsilon$, but by (4) we know that there exists a positive integer M such that for $k\geq M$, $\rho(x_{n_k-1}, x'_{n_k})<\varepsilon$. So let $N=n_k$ (for some $k\geq M$). Clearly for all $n\geq N$

$$\rho(x_n, x_{n+1})<\cdots\cdots<\rho(x_{N-1} x'_N)<\varepsilon$$

and this is a contradiction. Hence (2) follows. A similar argument establishes (3).

We shall now show that the sequence $\{x_n\}$ is Cauchy. Contruct a sequence $\{x_n\}$ such that $x_n=f^n x, x\in M$. Also we note that condition (1) implies that

(7) $\quad \rho(fx, fy)<\max\{\rho(x, y), \rho(x, fx), \rho(y, fy)\}, \quad x\neq y, \quad x, y\in M$

Then from (7) we derive that $\rho(x_{n+1}, x_{n+2})<\rho(x_n, x_{n+1})$. Hence $\rho(x_n, x_{n+1})$

decreases with n and, because of (1), $\rho(x_n, x_{n+1})\to 0$ as $n\to\infty$. Next we shall show that the sequence $\{x_n\}$ is fundamental. If it is not, there exists an $\varepsilon>0$ such that lim sup $\rho(x_m, x_n)>2\varepsilon$. Since $\rho(x_n, x_{n+1})\to 0$, there exists a positive integer N such that $\rho(x_j, x_{j+1})<\delta'/3$ for $j\geq N$, where $\delta'=\min(\delta, \varepsilon)$. Choose further two integers $m, n>N$ so that $\rho(x_m, x_n)>2\varepsilon>\varepsilon+\delta'$. This implies that there exists an integer k in [m, n] with

(8) $$\varepsilon+\frac{2\delta'}{3}<\rho(x_m, x_k)<\varepsilon+\delta'.$$

Now from the triangle inequality we obtain

(9) $$\rho(x_m, x_k)\leq\rho(x_m, x_{m+1})+\rho(x_{m+1}, x_{k+1})+\rho(x_{k+1}, x_k)$$
$$<\delta'/3+\varepsilon+\delta'/3=\varepsilon+\frac{2\delta'}{3}$$

since max $\{\rho(x_m, x_k), \rho(x_m, x_{m+1}), \rho(x_k, x_{k+1})\}=\rho(x_m, x_k)$. Then (9) contradicts (8) and hence $\{x_n\}$ is fundamental sequence. Let $u=\lim_n x_n$. Since $F(x)$ is lower semi-continuous, lim inf $F(x_n)\geq F(u)$. But $\rho(x_n, x_{n+1})\to 0$ as $n\to\infty$; thus $F(u)=0$ implying $f(u)=u$. Then u is a fixed point of f and because of (7), it is clearly unique.

NOTE The above result is patterned after the result of Assad [1] with necessary modifications as required for the more general settings.

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REFERENCES

[1] N. A. Assad, *A fixed point theorem for weakly uniformly strict contraction*, Canad. Math. Bull., 16, 15—18 (1973).

[2] N. A. Assad and W. A. Kirk, *Fixed point theorems for set valued mappings of contractive type*, Pacific J. Math., 43, 553—562 (1972).

[3] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., 28, 326—329 (1969).

[4] M. Maiti, J. Achari and T. K. Pal, *Remarks on some fixed point theorems*, Publ. Inst. Math., 21, 115—118 (1977).