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SOME INTEGRALS ASSOCIATED WITH THE H-FUNCTION OF SEVERAL COMPLEX VARIABLES

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1. Introduction

The H-function of several complex variables is defined and represented in the following manner (Srivastava and Panda, 1976):

(1.1) $$H\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = H\begin{matrix} M:(P',Q'):\ldots:(P^{(n)},Q^{(n)}) \\ A,C:(B',D'):\ldots:(B^{(n)},D^{(n)}) \end{matrix}\left[\begin{matrix} [(a):\theta',\ldots,\theta^{(n)}]: \\ [(c):\varepsilon',\ldots,\varepsilon^{(n)}]: \end{matrix}\right.$$

$$\left.\begin{matrix} [(b'):\phi']:\ldots:[(b^{(n)}):\phi^{(n)}]; \\ [(d'):\delta']:\ldots:[(D^{(n)}):\delta^{(n)}]; \end{matrix}\ z_1,\ldots,z_n\right]$$

$$=(2\pi i)^{-n}\int_{L_1}\ldots\int_{L} U_1(s_1)\ldots U_n(s_n)\ V(s_1,\ldots,s_n) z_1^{s_1}\cdots z_n^{s_n}\ ds_1\cdots ds_n,$$

where

(1.2) $$U_i(s_i)=\frac{\prod_{j=1}^{P^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}s_i]\prod_{j=1}^{Q^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}s_i]}{\prod_{j=P^{(i)}+1}^{D^{(i)}}\Gamma[1-d_j^{(i)}+\delta_j^{(i)}s_i]\prod_{j=Q^{(i)}+1}^{B^{(i)}}\Gamma[b_j^{(i)}-\phi_j^{(i)}s_i]}$$
$(i=1,\ldots,n)$

(1.3) $$V(s_1,\ldots,s_n)=\frac{\prod_{j=1}^{M}\Gamma[1-a_j+\sum_{i=1}^{n}\theta_j^{(i)}s_i]}{\prod_{j=M+1}^{A}\Gamma[a_j-\sum_{i=1}^{n}\theta_j^{(i)}s_i]\prod_{j=1}^{C}\Gamma[1-c_j+\sum_{i=1}^{n}\varepsilon_j^{(i)}s_i]},$$

an empty product is interpreted as unity, also $M, A, C, P^{(i)}, B^{(i)}, D^{(i)}$ are such that

$A\geq M\geq 0,\ C\geq 0,\ D^{(i)}\geq P^{(i)}\geq 0,\ B^{(i)}\geq Q^{(i)}\geq 0,$ and $\theta_j^{(i)}(j=1,\ldots,A)$, $\phi_j^{(i)}(j=1,\ldots,B^{(i)})$, $\varepsilon_j^{(i)}(j=1,\ldots,C)$, $\delta_j^{(i)}(j=1,\ldots,D^{(i)})$, $i=1,\ldots,n$, are positive quantities.

The contours L_i are defined suitable and all the poles of the integrand are assumed to be simple.

The multiple integral in (1.1) converges absolutely if

$$\left|\arg (z_i)\right| < \frac{1}{2}\pi T_i,\ i=1,\dots,n,$$

where

$$(1.4)\quad T_i=\sum_{j=1}^{M}\theta_j^{(i)}-\sum_{j=M+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{Q^{(i)}}\phi_j^{(i)}-\sum_{j=1+Q^{(i)}}^{B^{(i)}}\phi_j^{(i)}$$
$$-\sum_{j=1}^{C}\varepsilon_j^{(i)}+\sum_{j=1}^{P^{(i)}}\delta_j^{(i)}-\sum_{j=1+P^{(i)}}^{D^{(i)}}\delta_j^{(i)},\ i=1,\dots,n$$

(a) denotes the sequence of A parameters $a_1,\dots,a_A$; for each $i=1,\dots,n$, $(b^{(i)})$ abbreviates the sequence of $B^{(i)}$ parameters $b_j^{(i)}$ $(j=1,\dots,B^{(i)})$, with similar interpretation for $(c), (d^{(i)})$, etc.. Also $b^{(1)}=b'$, $b^{(2)}=b''$, and so on.

2. The definite integrals

We make use of the definition (1.1) and the known result [1, p. 698]

$$(2.1)\quad \int_{-1}^{1}(1+x)^{g-1}(1-x)^{t-1}P_w^{(u,v)}\left(1-\frac{my}{2}(1-x)\right)dx$$
$$=\frac{2^{g+t-1}(u+1\,;\,w)\Gamma(g)}{w!}\sum_{r=0}^{w}\frac{(-w\,;\,r)(1+u+v+w\,;\,r)\Gamma(t+r)}{r!(u+1\,;\,r)\Gamma(t+g+r)}\left(\frac{my}{2}\right)^r,$$

$\mathrm{Re}(g)>0$, $\mathrm{Re}(t)>0$, and we obtain

$$(2.2)\quad \int_{-1}^{1}(1+x)^{g-1}(1-x)^{t-1}P_w^{(u,v)}\left(1-\frac{my}{2}(1-x)\right)H\begin{bmatrix} z_1(1+x)^{h_1}(1-x)^{k_1} \\ \vdots \\ z_n(1+x)^{h_n}(1-x)^{k_n}\end{bmatrix}dx$$
$$=\frac{2^{g+t-1}(u+1\,;\,w)}{w!}\sum_{r=0}^{w}\frac{(-w\,;\,r)(1+u+v+w\,;\,r)(my/2)^r}{r!(u+1\,;\,r)}$$
$$\cdot H^{M+2\,:\,(P',Q')\,;\,\dots\,;\,(P^{(n)},Q^{(n)})}_{A+2,\,C+1\,:\,(B',D')\,;\,\dots\,;\,(B^{(n)},D^{(n)})}\left[\begin{matrix}[1-g:h_1,\dots,h_n],\ [1-t-r:k_1,\dots,k_n], \\ [1-t-g-r:h_1+k_1,\dots,h_n+k_n],\end{matrix}\right.$$
$$\left.\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\varepsilon',\dots,\varepsilon^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1 2^{h_1+k_1},\dots,z_n 2^{h_n+k_n}\right],$$

where $h_i>0, k_i>0, T_i>0$, $|\arg\ z_i|<\frac{1}{2}\pi T_i$, $\mathrm{Re}(g+\sum_{i=1}^{n} h_i d_j^{(i)}/\delta_j^{(i)})>0$, $\mathrm{Re}\ (t+\sum_{i=1}^{n} k_i d_j^{(i)}/\delta_j^{(i)})>0$, $i=1,\dots,n\,;\,j=1,\dots,P^{(i)}$ and the series on the right converges.

The second integral, which involve the Gauss hypergeometric function, follow similarly from the known result [3, p. 399]

$$(2.3)\quad \int_0^1 x^{w-1}(1-x)^{s-1}{}_2F_1(u,v\,;\,w\,;\,x)=\frac{\Gamma(w)\Gamma(s)\Gamma(w+s-u-v)}{\Gamma(w+s-u)\Gamma(w+s-v)},$$

$\mathrm{Re}(w)>0$, $\mathrm{Re}(s)>0$, $\mathrm{Re}(w+s-u-v)>0$, and we find

$$(2.4)\quad \int_0^1 x^{w-1}(1-x)^{s-1}{}_2F_1(u,v\,;\,w\,;\,x)\ H\begin{bmatrix} z_1(1-x)^{h_1} \\ \vdots \\ z_n(1-x)^{h_n}\end{bmatrix}dx$$

$$=\Gamma(w)\ H^{M+1\,:\,(P',Q')\,:\,\dots\,:\,(P^{(n)},\,Q^{(n)})}_{A+2,C+1\,:\,(B',D')\,:\,\dots\,:\,(B^{(n)},\,D^{(n)})}\left[\begin{matrix}[1-s:h_1,\dots,h_n],\ [1+u+v-w-s:h_1,\dots,h_n],\ [(a):\theta',\dots,\theta^{(n)}]:\\ [1+u-w-s\,;\,h_1,\dots,h_n],\ [1+v-w-s:h_1,\dots,h_n],\ [(c):\varepsilon',\dots,\varepsilon^{(n)}]:\end{matrix}\ \begin{matrix}[(b'):\phi']\,;\dots;\,[(b^{(n)}):\phi^{(n)}]\,;\\ [(d'):\delta']\,;\dots;\,[(d^{(n)}):\delta^{(n)}]\,;\end{matrix}\ z_1,\dots,z_n\right],$$

where $\mathrm{Re}(w)>0, \mathrm{Re}(s+\sum_{i=1}^{n} h_i d_j^{(i)}/\delta_j^{(i)})>0$, $h_i>0, T_i>0$,

$|\arg\ z_i|<\frac{1}{2}\pi T_i$, $\mathrm{Re}(w+s-u-v)>0$, $i=1,\dots,n\,;\,i=1,\dots,P^{(i)}$.

In a similar manner, by applying the formula

$$(2.5)\quad \int_0^t x^{w-1}(t-x)^{s-1}dx=\frac{\Gamma(w)\Gamma(s)}{\Gamma(w+s)}t^{w+s-1}, \mathrm{Re}(w)>0,\ \mathrm{Re}(s)>0.$$

$$(2.6)\quad \int_0^t x^{w-1}(t-x)^{s-1}H\begin{bmatrix} z_1x^{h_1}(t-x)^{k_1} \\ \vdots \\ z_nx^{h_n}(t-x)^{k_n}\end{bmatrix}dx$$

$$=t^{w+s-1}\ H^{M+2\,:\,(P',Q')\,:\,\dots\,:\,(P^{(n)},Q^{(n)})}_{A+2,C+1\,:\,(B',D')\,:\,\dots\,:\,(B^{(n)},D^{(n)})}\left[\begin{matrix}[1-w:h_1,\dots,h_n],\ [1-s:k_1,\dots,k_n],\ [(a):\theta',\dots,\theta^{(n)}]:\\ [1-w-s:h_1+k_1,\dots,h_n+k_n],\ [(c):\varepsilon',\dots,\varepsilon^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi']:\ldots:[(b^{(n)}),\ \phi^{(n)}]: \\ [(d'):\delta']:\ldots:[(d^{(n)}),\delta^{(n)}]:\end{matrix}\ z_1 t^{h_1+k_1},\ldots,z_n\ t^{h_n+k_n}\right],$$

where $h_i>0,\ k_i>0,\ T_i>0,\ |\arg\ z_i|<\frac{1}{2}\pi T_i,\quad i=1,\ldots,n;\ j=1,\ldots,P^{(i)}$,

$\mathrm{Re}(w+\sum_{i=1}^{n} h_i d_j^{(i)}/\delta_j^{(i)})>0,\ \mathrm{Re}(s+\sum_{i=1}^{n} k_i d_j^{(i)}/\delta_j^{(i)})>0.$

Particular cases:

(i) If $m=2,\ y=1,\ t=\sigma+1,\ g=\beta+1$ in (2.2), and making use of Saalchutz's theorem [4], we get a result recently obtained by Srivastava and Panda [6, p. 131(2.2)].

(ii) For $n=2$, the integral formulae in (2.2), (2.4) and (2.6) reduces to results recently obtained by Chaurasia [2].

(iii) The results in (2.2), (2.4) and (2.6) can be deduced for the generalized Lauricella functions etc. [7] in view of a formula [6, p. 139(4.11)].

A great number of interesting integral formulae as particular cases of our main results can be deduced, but we omit them here for lack of space.

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