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NOTE ON ANTI-HOLOMORPHIC SUBMANIFOLDS OF REAL CODIMENSION OF A COMPLEX PROJECTIVE SPACE

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1. Introduction

As is well known, the unit hypersurface $S^{2m+1}(1)$ in an $(m+1)$-dimensional complex number space C^{m+1}, which will be naturally identified with $R^{2(m+1)}$, is a principal circle bundle over a complex projective space CP^m and the Riemannian structure on CP^m is given by $\tilde{\pi}$: $S^{2m+1}(1)\longrightarrow CP^m$ the natural projection of $S^{2m+1}(1)$ onto CP^m which is defined by the Hopf-fibration (see [2], [6]). Thus the theory of submersion is used as an interesting tool for studying a complex projective space and its submanifolds. For example, Lawson [1] introduced the notion of generalized equators $M^C_{q,s}(a,b)$ and Maeda [3], Okumura [4] and etc. have determined necessary or necessary and sufficient conditions for real hypersurfaces to be one of the model spaces $M^C_{q,s}(a,b)$.

On the other hand a submanifold M of a Kaehlerian manifold is called a *generic submanifold* (an anti-holomorphic submanifold) if the normal space $N_p(M)$ of M at P is always mapped into the tangent space $T_p(M)$ of M at P under the action of the almost complex structure tensor ϕ of the ambient manifold, that is, if $\phi N_p(M)\subset T_p(M)$ for all $P\in M$ (see [5], [7] and [9]).

In [9], Yano and Kon gave some examples of generic submanifolds immersed in complex space forms and found the characterizations of the examples by using the method of Riemannian fibre bundles.

The purpose of the present paper is to study generic submanifolds of CP^m by the method of Riemannian fibre bundles and give the characterization of a generic model immersed in CP^m by using the following theorem.

THEOREM A (Yano and Kon [9]) *Let M be a complete minimal submanifold of dimension n immersed in an $(n+p)$-dimensional unit sphere $S^{n+p}(1)$ with parallel second fundamental form. If the square of length of the second fundamental form is not smaller than pn, then M is a pythagorian product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),\ r_t=\sqrt{p_t/n}\ (t=1,\cdots,N),$$

where $p_1,\cdots,p_N\geq 1,\ p_1+\cdots+p_N=n,\ p=N-1.$

Manifolds, submanifolds, geometric objects and mappings we discuss in this paper are assumed to be differentiable and of class C^∞. We use in the present paper the systems of indices as follows:

$$\kappa,\mu,\nu,\lambda=1,2,\cdots,2m+1;\ h,i,j,k=1,2,\cdots,2m,$$
$$\alpha,\beta.\gamma,\delta=1,2,\cdots,n+1;\ a,b,c,d,e=1,2,\cdots,n,$$
$$x,y,z,\ w=1,2,\cdots,p,\ n+p=2m.$$

The summation convention will be used with respect to those systems of indices.

2. Submanifolds of Kaehlerian manifolds

Let $\tilde{M}$ be a $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{\tilde{U};y^j\}$ and denote by g_{ji} components of the Hermitian metric tensor and by ϕ_i^j those of the almost complex structure of M. Then we have

$$\phi_h^{\ i}\phi_j^{\ h}=-\delta_j^{\ i}, \tag{2.1}$$

$$\phi_j^{\ h}\phi_i^{\ k}g_{hk}=g_{ji}, \tag{2.2}$$

and denoting by $\tilde{\nabla}_j$ the operator of covariant differentiation with respect to g_{ji},

$$\tilde{\nabla}_j\phi_i^{\ h}=0. \tag{2.3}$$

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U;\ x^a\}$ and immersed isometrically in $\tilde{M}$ by the immersion $i: M\longrightarrow\tilde{M}$. In the sequel we identify $i(M)$ with M itself and represent the immersion by

$$y^j=y^j(x^c). \tag{2.4}$$

We put

$$B_b^{\ j}=\partial_b y^j,\ \partial_b=\partial/\partial x^b \tag{2.5}$$

and denote by $N_x^{\ h}$ mutually orthogonal unit normals to M. Then denoting by g_{cb} the fundamental metric tensor of M, we have

$$g_{cb}=B_c^{\ j}B_b^{\ i}g_{ji}$$

becauce the immersion is isometric. Therefore, denoting by ∇_b the operator of van der Waerden-Bortolotti covariant differentiation with respect to g_{cb}, we have equations of Gauss and Weingarten for M

$$\nabla_c B_b^j = A_{cb}{}^x N_x^j, \tag{2.6}$$

$$\nabla_c N_x^j = -A_c{}^b{}_x B_b^j, \tag{2.7}$$

respectively, where $A_{cb}{}^x$ are the second fundamental tensors with respect to the normals N_x^j and $A_{cx}{}^a = A_{cax} g^{ab} = A_{ca}{}^y g^{ab} g_{xy}$, g_{xy} being the metric tensor of the normal bundle of M give by $g_{xy} = N_x^j N_y^i g_{ji}$ and $(g^{ba}) = (g_{ba})^{-1}$.

Equations of Gauss, Codazzi and Ricci are respectively given by

$$K_{dcb}{}^a = K_{kji}{}^h B_{dcbh}{}^{kjia} + A_d{}^a{}_x A_{cb}{}^x - A_c{}^a{}_x A_{db}{}^x, \tag{2.8}$$

$$0 = K_{kji}{}^h B_{dcb}{}^{kji} N_h^x - (\nabla_d A_{cb}{}^x - \nabla_c A_{db}{}^x). \tag{2.9}$$

and

$$K_{dcy}{}^x = K_{kji}{}^h B_{dc}{}^{kj} N_y^i N_h^x + (A_{de}{}^x A_c{}^e{}_y - A_{ce}{}^x A_d{}^e{}_y), \tag{2.10}$$

where $B_{dcbh}{}^{kjia} = B_d^k B_c^j B_b^i B_h^a$, $B_{dcb}{}^{kji} = B_d^k B_c^j B_b^i$, $B_h^a = B_b^j g^{ba} g_{jh}$, $N_h^x = N_y^j g^{yx} g_{jh}$ and $K_{dcy}{}^x$ is the curvature tensor of the connection induced in the normal bundle.

We now consider the transforms $\phi_i^j B_b^i$ and $\phi_i^j N_x^i$ of B_b^i and N_x^i by the structure tensor ϕ_i^j. Then we can put in each coordinate neighborhood $U = \tilde{U} \cap M$

$$\phi_i^j B_b^i = \phi_b^a B_a^j + \phi_b^x N_x^j, \tag{2.11}$$

$$\phi_i^j N_x^i = -\phi_x^a B_a^j + \phi^y{}_x N_y^j \tag{2.12}$$

respectively.

Using $\phi_{ji} = -\phi_{ij}$, $\phi_{ji} = \phi_j^h g_{hi}$, we have, from (2.11) and (2.12),

$$\phi_{bx} = \phi_{xb}, \tag{2.13}$$

where $\phi_{bx} = \phi^y{}_b g_{yx}$ and $\phi_{xb} = \phi_x^a g_{ab}$, and

$$\phi_{yx} = -\phi_{xy}, \tag{2.14}$$

where $\phi_{yx} = \phi_y^z g_{zx}$.

Applying ϕ to (2.11) and (2.12) and using (2.1) and these equations, we can easily find

(2.15) $\phi_a^{\ b}\phi_b^{\ c}+\delta_a^{\ c}=\phi_a^{\ x}\phi_x^{\ c}$,

(2.16) $\phi_a^{\ b}\phi_b^{\ y}+\phi_a^{\ x}\phi_x^{\ y}=0,\ \phi_x^{\ a}\phi_a^{\ b}+\phi_x^{\ y}\phi_y^{\ b}=0$,

(2.17) $\phi_x^{\ z}\phi_z^{\ y}+\delta_x^{\ y}=\phi_x^{\ a}\phi_a^{\ y}$

Differentiating (2.11) and (2.12) covariantly along M and using (2.3), (2.6) and (2.7), we can verify that

(2.18) $\nabla_b\phi_a^{\ c}=A_b^{\ c}{}_x\phi_a^{\ x}-A_{ba}^{\ \ x}\phi_x^{\ c}$,

(2.19) $\nabla_b\phi_a^{\ x}=A_{ba}^{\ \ y}\phi_y^{\ x}-A_{bc}^{\ \ x}\phi_a^{\ c},\ \nabla_b\phi_x^{\ a}=A_b^{\ c}{}_x\phi_c^{\ a}-A_b^{\ a}{}_y\phi_x^{\ y}$,

(2.20) $\nabla_b\phi_x^{\ y}=A_{ba}^{\ \ y}\phi_x^{\ a}-A_b^{\ a}{}_x\phi_a^{\ y}$.

we now assume that the ambient manifold $\tilde{M}$ is of constant holomorphic sectional curvature c. Then it is well known that its curvature tensor $K_{kji}^{\ \ \ h}$ has the form

(2.21) $K_{kji}^{\ \ \ h}=\frac{c}{4}(\delta_k^{\ h}g_{ji}+\delta_j^{\ h}g_{ki}+\phi_k^{\ h}\phi_{ji}-\phi_j^{\ h}\phi_{ki}-2\phi_{kj}\phi_i^{\ h})$.

Therefore, substituting (2.21) into (2.8), (2.9) and (2.10), we can see that the equations of Gauss, Codazzi and Ricci are respectively given by

(2.22) $K_{dcb}^{\ \ \ a}=\frac{c}{4}(\delta_d^{\ a}g_{cb}-\delta_c^{\ a}g_{db}+\phi_d^{\ a}\phi_{cb}-\phi_c^{\ a}\phi_{db}-2\phi_{dc}\phi_b^{\ a})+A_d^{\ a}{}_xA_{cb}^{\ \ x}-A_c^{\ a}{}_xA_{db}^{\ \ x}$,

(2.23) $\nabla_dA_{cb}^{\ \ x}-\nabla_cA_{db}^{\ \ x}=\frac{c}{4}(\phi_d^{\ x}\phi_{cb}-\phi_c^{\ x}\phi_{db}-2\phi_{dc}\phi_b^{\ x})$,

(2.24) $K_{dcy}^{\ \ \ x}=\frac{c}{4}(\phi_d^{\ x}\phi_{cy}-\phi_c^{\ x}\phi_{dy}-2\phi_{dc}\phi_y^{\ x})+A_{de}^{\ \ x}A_c^{\ e}{}_y-A_{ce}^{\ \ x}A_d^{\ e}{}_y$.

3. Submersion $\tilde{\pi}: S^{2m+1}\longrightarrow CP^m$ and immersion $i: M\longrightarrow CP^m$

Let S^{2m+1} (1) be the hypersphere $\{(c^1,\cdots,c^{m+1})|\ |c^1|^2+\cdots+|c^{m+1}|^2=1\}$ of radius 1 in the $(m+1)$-dimensional complex space C^{m+1} which will be identified naturally with $R^{2(m+1)}$. The sphere S^{2m+1} (1) will be simply denoted by S^{2m+1}.

Let $\tilde{\pi}: S^{2m+1}\longrightarrow CP^m$ be the natural projective of S^{2m+1} onto a complex projective space CP^m which is defined by the Hopf-fibration. We consider a Riemannian submersion $\pi:\overline{M}\longrightarrow M$ compatible with the Hopf-fibration $\tilde{\pi}$: $S^{2m+1}\longrightarrow CP^m$, where M is a submanifold of codimension p in CP^m and $\overline{M}=\pi^{-1}(M)$ that of S^{2m+1}. More precisely speaking, $\pi:\overline{M}\longrightarrow M$ is a Riemannian submersion with totally goedesic fibres such that the following diagram commutative:

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\ \bar{i}\ } & S^{2m+1} \\ \pi\downarrow & & \downarrow\tilde{\pi} \\ M & \xrightarrow[\ i\ ]{} & CP^m \end{array}$$

where $\bar{i}:\overline{M}\longrightarrow S^{2m+1}$ and $i:M\longrightarrow CP^m$ are certain isometric immersions.

Covering S^{2m+1} by a system of coordinate neighborhoods $\{\hat{U}:y^\kappa\}$ such that $\tilde{\pi}(\hat{U})=\tilde{U}$ are coodinate neighborhoods of CP^m with local coordinate (y^j), we represent the projection $\tilde{\pi}:S^{2m+1}\longrightarrow CP^m$ by

(3.1) $$y^j=y^j(y^\kappa)$$

and put

(3.2) $$E_\kappa{}^j=\partial_\kappa y^j,\ \ \partial_\kappa=\partial/\partial y^\kappa,$$

the rank of matric (E_k^j) being always $2m$.

Let's denote by $\tilde{\xi}^\kappa$ components of the unit Sasakian structure vector in S^{2m+1}. Since the unit vector field is always tangent to the fibre $\tilde{\pi}^{-1}(\tilde{p})$, $\tilde{p}\in CP^m$ everywhere, E_k^j and $\tilde{\xi}_\kappa$ from a local corame in S^{2m+1}, where $\tilde{\xi}_\kappa=g_{\kappa\mu}\tilde{\xi}^\mu$ and $g_{\kappa\mu}$ denote the Riemannian metric tensor of S^{2m+1}. We denote by $\{E_j^\kappa,\ \tilde{\xi}^\kappa\}$ the frame corresponding to the coframe $\{E_\kappa^j, \tilde{\xi}_\kappa\}$. We then have

(3.3) $$E_\kappa{}^jE_i{}^\kappa=\delta_i^j,\ \ E_\kappa{}^j\tilde{\xi}^\kappa=0,\ \ \tilde{\xi}_\kappa E_i{}^k=0.$$

We now take coordinate neihgborhoods $\{\overline{U}:x^\alpha\}$ of $\overline{M}$ such that $\pi(\overline{U})=U$ are coordinate neigborhoods of M with local coordinates (x^a). Let the isometric immersions $\bar{i}$ and i be locally expressed by $y^\kappa=y^\kappa(x^\alpha)$ and $y^j(x^a)$ in terms of local coordinates x^α in $\overline{U}$ $(\subset\overline{M})$ and (x^a) in U $(\subset M)$ respectively. Then the commutativity $\tilde{\pi}\cdot\bar{i}=i\cdot\pi$ of the diagram implies

$$y^j(x^a(x^\alpha))=y^j(y^\kappa(x^\alpha)),$$

where we expressed the submersion by $x^a=x^a(x^\alpha)$ locally, and hence

(3.4) $$B_a{}^jE_\alpha{}^a=E_\kappa{}^jB_\alpha{}^\kappa,$$

$B_a{}^j=\partial_a y^j$, $B_\alpha{}^\kappa=\partial_\alpha y^\kappa$ and $E_\alpha{}^a=\partial_\alpha x^a$.

For an arbitrary point $p\in M$ we choose unit normal voctor fields $N_x{}^j$ to M defined in a neighborhood U of p in such a way that $\{B_a{}^j, N_x{}^j\}$ span the tangent space of CP^m at $i(p)$. Let $\bar{p}$ be an arbitrary point of the fibre $\pi^{-1}(p)$ over p, then the lifts $N_x^{\ \kappa}=N_x^{\ j}E_j^{\ \kappa}$ of $N_x^{\ j}$ are unit normal vector fields to $\overline{M}$ defined in the tubular neighborhood over U because of (3.4). Since $\tilde{\xi}^\kappa E_\kappa^{\ j}=0$, we can represent $\tilde{\xi}$ by

$$\tilde{\xi}^\kappa=\xi^\alpha B_\alpha^{\ \kappa}, \tag{3.5}$$

where ξ^α is a local vector field in $\overline{M}$. Using (3.4) and (3.5), we find

$$\xi_\alpha\xi^\alpha=1,\ \ \xi^\alpha E_\alpha^{\ a}=0, \tag{3.6}$$

where $\xi_\alpha=\xi^\beta g_{\beta\alpha}$ and $g_{\beta\alpha}$ is the Riemannian metric tensor of $\overline{M}$ induced from that of S^{2m+1}. Therefore, $\{E_\alpha^{\ a},\ \xi_\alpha\}$ is a local coframe in $\overline{M}$ corresponding to $\{E_\kappa^j, \tilde{\xi}_\kappa\}$ in S^{2m+1}. Denoting by $\{E_a^\alpha, \xi^\alpha\}$ the frame corresponding to this coframe $\{E_\alpha^a, \xi_\alpha\}$ we have

$$E_\alpha^{\ b}E_a^{\ \alpha}=\delta_a^{\ b},\ \ \xi_\alpha E_b^{\ \alpha}=0, \tag{3.7}$$

and consequently

$$E_j^{\ \kappa}B_b^{\ j}=B_\alpha^{\ \kappa}E_b^{\ \alpha} \tag{3.8}$$

with the help of (3.4) and (3.6).

Denoting by $\left\{ {\lambda \atop \mu\ \nu} \right\}$, $\left\{ {i \atop j\ h} \right\}$, $\left\{ {\alpha \atop \beta\ \gamma} \right\}$ and $\left\{ {a \atop b\ c} \right\}$ the Christoffel symbols formed with the Riemannian metrics $g_{\mu\lambda}$, g_{ji}, $g_{\beta\alpha}$ and g_{ba} respectively, we put

$$D_\mu E_\lambda^{\ i}=\partial_\mu E_\lambda^{\ i}-\left\{ {\kappa \atop \mu\lambda} \right\}E_\kappa^{\ i}+\left\{ {i \atop jh} \right\}E_\mu^{\ j}E_\lambda^{\ h},$$

$$D_\mu E_i^{\ \lambda}=\partial_\mu E_i^{\ \lambda}+\left\{ {\lambda \atop \mu\kappa} \right\}E_i^{\ \kappa}-\left\{ {h \atop ji} \right\}E_\mu^{\ j}E_h^{\ \lambda},$$

and

$$\overline{\nabla}_\beta E_\alpha^a=\partial_\beta E_\alpha^a-\left\{ {\gamma \atop \beta\alpha} \right\}E_\gamma^a+\left\{ {a \atop bc} \right\}E_\beta^b E_\alpha^c,$$

$$\overline{\nabla}_\beta E_a^{\ \alpha}=\partial_\beta E_a^{\ \alpha}+\left\{ {\alpha \atop \beta\gamma} \right\}E_a^{\ \gamma}-\left\{ {c \atop ba} \right\}E_\beta^{\ b}E_c^{\ \alpha}.$$

Since the metrics $g_{\mu\lambda}$ and $g_{\beta\alpha}$ are both invariant with respect to the submersions $\tilde{\pi}$ and π respectively, the van der Waerden-Bortolotti covariant derivatives of $E_\lambda^{\ i}$, $E_i^{\ \lambda}$ and $E_\alpha^{\ a}$, $E_a^{\ \alpha}$ are given by

$$\begin{cases} D_\mu E_\lambda^{\ i}=h_j^{\ i}(E_\mu^{\ j}\tilde{\xi}_\lambda+\tilde{\xi}_\mu E_\lambda^{\ j}), \\ D_\mu E_i^{\ \lambda}=h_{ji}E_\mu^{\ j}\tilde{\xi}^\lambda-h_i^{\ j}\tilde{\xi}_\mu E_j^{\ \lambda}, \end{cases} \tag{3.9}$$

(3.10)
$$\begin{cases}\overline{\nabla}_\beta E_\alpha^{\ a}=h_b^{\ a}(E_\beta^{\ b}\xi_\alpha+\xi_\beta E_\alpha^{\ b}),\\ \overline{\nabla}_\beta E_a^{\ \alpha}=h_{ba}E_\beta^{\ b}\xi^\alpha-h_a^{\ b}\xi_\beta E_b^{\ a}\end{cases}$$

respectively, where $h_j^{\ h}=g^{ih}h_{ji}$, $h_b^{\ a}=g^{ac}h_{bc}$, h_{ji} and h_{ba} being the structure tensors induced from the submersions $\tilde{\pi}$ and π respectively (See Ishihara and Konishi [2]).

On the other side, the equations of Gauss and Weingarten for the immersion $\tilde{i}: M\longrightarrow S^{2m+1}$ are given by

(3.11)
$$\begin{cases}\overline{\nabla}_\beta B_\alpha^{\ \kappa}=\partial_\beta B_\alpha^{\ \kappa}+\left\{{\kappa \atop \mu\lambda}\right\}B_\beta^{\ \mu}B_\alpha^{\ \lambda}-\left\{{\gamma \atop \beta\alpha}\right\}B_\gamma^{\ \kappa}=A_{\beta\alpha}^{\ \ x}N_x^{\ \kappa},\\ \overline{\nabla}_\beta N_x^{\ \kappa}=\partial_\beta N_x^{\ \kappa}+\left\{{\kappa \atop \mu\lambda}\right\}B_\beta^{\ \mu}N_x^{\ \lambda}-\Gamma_{\beta\ x}^{\ y}N_y^{\ \kappa}=-A_{\beta\ x}^{\ \alpha}B_\alpha^{\ \kappa},\end{cases}$$

and those for the immersion $i: M\longrightarrow CP^m$ by

(3.12)
$$\begin{cases}\nabla_b B_a^{\ i}=\partial_b B_a^{\ i}+\left\{{i \atop jh}\right\}B_b^{\ j}B_a^{\ h}-\left\{{c \atop ba}\right\}B_c^{\ i}=A_{ba}^{\ \ x}N_x^{\ i},\\ \nabla_b N_x^{\ i}=\partial_b N_x^{\ i}+\left\{{i \atop jh}\right\}B_b^{\ j}N_x^{\ h}-\Gamma_{b\ x}^{\ y}N_y^{\ i}=-A_{b\ x}^{\ a}B_a^{\ i},\end{cases}$$

$\Gamma_{\beta\ x}^{\ y}$ and $\Gamma_{b\ x}^{\ y}$ being components of the connections induced on the normal bundles $N(\overline{M})$ and $N(M)$ of $\overline{M}$ and M respectively, where $A_{\beta\ x}^{\ \alpha}=A_{\beta\gamma}^{\ \ y}g^{\gamma\alpha}g_{yx}$, $A_{\beta\alpha}^{\ \ x}$ and $A_{ba}^{\ \ x}$ are the second fundamental tensors of $\overline{M}$ and M with respect to the unit normals $N_x^{\ \kappa}$ and $N_x^{\ j}$ respectively. Moreover in such a case (3.4) and (3.8) imply

$$\nabla_b=E_b^{\ \alpha}\overline{\nabla}_\alpha.$$

We now put $\phi_\mu^{\ \lambda}=D_\mu\tilde{\xi}^\lambda$. Then we have by definition of Sasakian structure

(3.13)
$$\phi_\mu^{\ \lambda}\phi_\kappa^{\ \mu}=-\delta_\kappa^{\ \lambda}+\tilde{\xi}_\kappa\tilde{\xi}^\lambda,\ \ \phi_\mu^{\ \lambda}\tilde{\xi}^\mu=0,\ \ \tilde{\xi}_\lambda\phi_\mu^{\ \lambda}=0,$$
$$\phi_{\mu\lambda}+\phi_{\lambda\mu}=0$$

and

(3.14)
$$D_\mu\phi_\lambda^{\ \kappa}=\tilde{\xi}_\lambda\delta_\mu^{\ \kappa}-\tilde{\xi}^\kappa g_{\mu\lambda},\ \ D_\mu\tilde{\xi}^\kappa=\phi_\mu^{\ \kappa},$$

where $\phi_{\mu\lambda}=g_{\kappa\lambda}\phi_\mu^{\ \kappa}$, Denoting by $\mathcal{L}$ the Lie differentiation with respect to the vector field $\tilde{\xi}$, we find

(3.15)
$$\mathcal{L}\phi_\mu^{\ \lambda}=0.$$

putting in each U

(3.16)
$$\phi_j^{\ i}=\phi_\mu^{\ \lambda}E_j^{\ \mu}E_\lambda^{\ i},$$

we can see that $\phi_j^{\ i}$ defines a global tensor field of the same type as that of $\phi_j^{\ i}$, which will be denoted by the same letter, with the help of (3.15), $\mathcal{L}E_j^{\ \mu}=0$ and $\mathcal{L}E_\lambda^{\ i}=0$. Moreover, using (3.9), (3,14) and (3.16), we easily see

$$\phi_j^{\ i}=-h_j^{\ i}, \tag{3.17}$$

which satisfies

$$\phi_j^{\ i}\phi_h^{\ j}=-\delta_h^{\ i}. \tag{3.18}$$

Differentiating (3.16) covariantly along CP^m and using (3.9) and (3.14), we have

$$\tilde{\nabla}_j\phi_i^{\ h}=0, \tag{3.19}$$

where $\tilde{\nabla}$ denotes the projection of D. Hence the base space CP^m admits a Kaehlerian structure $\{\phi_j^{\ i},\ g_{ji}\}$ which is represented by the structure tensor $h_j^{\ i}$ of the submersion $\tilde{\pi}: S^{2m+1}\longrightarrow CP^m$ defined by the Hopf fibration.

Let's denote by $K_{\kappa\mu\nu}^{\ \ \ \ \lambda}$ and $K_{kji}^{\ \ \ h}$ components of the curvature tensors of $(S^{2m+1}, g_{\mu\lambda})$ and (CP^m, g_{ji}) respectively. Since the unit sphere S^{2m+1} is a space of constant curvature 1, using the equations of co-Gauss, we have

$$K_{kji}^{\ \ \ h}=K_{\kappa\mu\nu}^{\ \ \ \ \lambda}E_k^{\ \kappa}E_j^{\ \mu}E_i^{\ \nu}E_\lambda^{\ h}+h_k^{\ h}h_{ji}-h_j^{\ h}h_{ki}-2h_{kj}h_i^{\ h}$$

and together with (3.17)

$$K_{kji}^{\ \ \ h}=\delta_k^{\ h}g_{ji}-\delta_j^{\ h}g_{ki}+\phi_k^{\ h}\phi_{ji}-\phi_j^{\ h}\phi_{ki}-2\phi_{kj}\phi_i^{\ h}.$$

Hence CP^m is a Kaehlerian manifold with constant holomorphic sectional curvature 4 (Cf. Ishihara and Konishi [2]).

Putting

$$\begin{cases}\phi_i^{\ j}B_b^{\ i}=\phi_b^{\ a}B_a^{\ j}+\phi_b^{\ x}N_x^{\ j},\\ \phi_i^{\ j}N_x^{\ i}=-\phi_x^{\ a}B_a^{\ j}+\phi_x^{\ y}N_y^{\ j},\end{cases} \tag{3.20}$$

as already shown in section 2, we can easily find the algebraic relation (2.13) ~(2.17) and the structure equations (2.18) (2.24) with $c=4$ which will be very useful.

Now we put in each neighborhood $\bar{U}$ of $\bar{M}$

$$\phi_\beta^{\ \alpha}=\phi_b^{\ a}E_\beta^{\ b}E_a^{\ \alpha},\ \ \phi_x^{\ \alpha}=\phi_x^{\ a}E_a^{\ \alpha},\ \ \phi_\alpha^{\ x}=\phi_a^{\ x}E_\alpha^{\ a}, \tag{3.21}$$

where, here and in the sequel, we denote the lifts of functions by the same letters as those the given functions. Then, using (3.4), (3.8), (3.20) and (3.21) and taking account of $N_x^{\ \kappa}=N_x^{\ j}E_j^{\ \kappa}$, we obtain

$$\phi_\mu^{\ \kappa}B_\alpha^{\ \mu}=\phi_\alpha^{\ \beta}B_\beta^{\ \kappa}+\phi_\alpha^{\ x}N_x^{\ \kappa}, \tag{3.22}$$

$$\phi_\mu^{\ \kappa}N_x^{\ \mu}=-\phi_x^{\ \alpha}B_\alpha^{\ \kappa}+\phi_x^{\ y}N_y^{\ \kappa}. \tag{3.23}$$

Transvecting $\phi_\kappa^{\ \lambda}$ to (3.22) and (3.23) respectively and using (3.13), (3.22) and (3.23) in the usual way, we can easily obtain that

$$\begin{aligned}&\phi_\alpha^{\ \gamma}\phi_\gamma^{\ \beta}-\phi_\alpha^{\ x}\phi_x^{\ \beta}-\xi_\alpha\xi^\beta=-\delta_\alpha^{\ \beta},\ \phi_\alpha^{\ \beta}\phi_\beta^{\ x}+\phi_\alpha^{\ y}\phi_y^{\ x}=0,\\ &\phi_x^{\ \beta}\phi_\beta^{\ \alpha}+\phi_x^{\ y}\phi_y^{\ \alpha}=0,\ \phi_x^{\ z}\phi_z^{\ y}-\phi_x^{\ \alpha}\phi_\alpha^{\ y}=-\delta_x^{\ y},\\ &\phi_\alpha^{\ \beta}\xi_\beta=0,\ \xi^\alpha\phi_\alpha^{\ \beta}=0,\ \phi_\alpha^{\ x}\xi^\alpha=0,\ \xi_\alpha\phi_x^{\ \alpha}=0,\\ &\phi_{\beta\alpha}=-\phi_{\alpha\beta},\ \phi_{\alpha x}=\phi_{x\alpha},\ \phi_{xy}=-\phi_{yx},\end{aligned} \tag{3.24}$$

where we have put $\phi_{\beta\alpha}=\phi_\beta^{\ \gamma}g_{\gamma\alpha}$, $\phi_{\alpha x}=\phi_\alpha^{\ y}g_{yx}$, $\phi_{x\alpha}=\phi_x^{\ \beta}g_{\beta\alpha}$ and $\phi_{xy}=\phi_x^{\ z}g_{zy}$. Applying the operator $\overline{\nabla}_\gamma=B_\gamma^{\ \kappa}D_\kappa$ to (3.22) and (3.23) respectively and making use of (3.11), (3.14), (3.22) and (3.23), we also find

$$\begin{aligned}&\overline{\nabla}_\gamma\phi_\beta^{\ \alpha}=\xi_\beta\delta_\gamma^{\ \alpha}-\xi^\alpha g_{\gamma\beta}+A_\gamma^{\ \alpha}{}_x\phi_\beta^{\ x}-A_{\gamma\beta}^{\ \ x}\phi_x^{\ \alpha},\\ &\overline{\nabla}_\beta\phi_\alpha^{\ x}=A_{\beta\alpha}^{\ \ y}\phi_y^{\ x}-A_{\beta\gamma}^{\ \ x}\phi_\alpha^{\ \gamma},\ \overline{\nabla}_\beta\phi_x^{\ \alpha}=A_\beta^{\ \gamma}{}_x\phi_\gamma^{\ \alpha}-A_\beta^{\ \alpha}{}_y\phi_x^{\ y},\\ &\overline{\nabla}_\beta\phi_x^{\ y}=A_{\beta\alpha}^{\ \ y}\phi_x^{\ \alpha}-A_\beta^{\ \alpha}{}_x\phi_\alpha^{\ y},\end{aligned} \tag{3.25}$$

Also, applying the operator $\overline{\nabla}_\beta$ to (3.5) and taking account of (3.11) and (3.14), we have

$$\overline{\nabla}_\beta\xi^\alpha=\phi_\beta^{\ \alpha},\ \xi^\alpha A_{\beta\alpha}^{\ \ x}=\phi_\beta^{\ x},\ A_\beta^{\ \alpha}{}_x\xi^\beta=\phi_x^{\ \alpha}, \tag{3.26}$$

which and (3.9) and (3.21) imply

$$\phi_b^{\ a}=-h_b^{\ a}. \tag{3.27}$$

Moreover, in such a submanifold $\overline{M}$, its Ricci equation is given by

$$K_{\beta\alpha y}^{\ \ \ \ x}=A_{\beta\gamma}^{\ \ x}A_\alpha^{\ \gamma}{}_y-A_{\alpha\gamma}^{\ \ x}A_\beta^{\ \gamma}{}_y \tag{3.28}$$

because the ambient manifold S^{2m+1} is a of space constant curvatre.
Now we apply the operator $\nabla_b=B_b^{\ j}\widetilde{\nabla}_j=E_b^{\ \alpha}\overline{\nabla}_\alpha$ to (3.4). Then, using (3.11) and (3.12), we have

$$A_{ba}^{\ \ x}N_x^{\ j}E_\alpha^{\ a}+B_a^{\ j}E_b^{\ \beta}\overline{\nabla}_\beta E_\alpha^{\ a}=B_b^{\ i}E_i^{\ \mu}(D_\mu D_\kappa^{\ j})B_\alpha^{\ \kappa}+E_\kappa^{\ j}E_b^{\ \beta}A_{\beta\alpha}^{\ \ x}N_x^{\ \kappa},$$

from which, taking account of (3.9), (3.10) and (3.27),

$$A_{ba}{}^{x}N_{x}{}^{j}E_{\alpha}{}^{a}-\phi_{b}{}^{a}B_{a}{}^{j}\xi_{\alpha}=-\phi_{i}{}^{j}B_{b}{}^{i}\xi_{\alpha}+(A_{\beta\alpha}{}^{x}E_{b}{}^{\beta})N_{x}{}^{j},$$

or using (3.20),

$$A_{\beta\alpha}{}^{x}E_{b}{}^{\beta}=A_{ba}{}^{x}E_{\alpha}{}^{a}+\phi_{b}{}^{x}\xi_{\alpha}. \tag{3.29}$$

Transvecting (3.29) with $E_{\gamma}{}^{b}$ and changing the index γ with β, we get

$$A_{\beta\alpha}{}^{x}=A_{ba}{}^{x}E_{\beta}{}^{b}E_{\alpha}{}^{a}+\xi_{\beta}\phi_{\alpha}{}^{x}+\xi_{\alpha}\phi_{\beta}{}^{x} \tag{3.30}$$

with the help of (3.21) and (3.26), from which, taking account of (3.7) and (3.24),

$$g^{\beta\alpha}A_{\beta\alpha}{}^{x}=g^{ba}A_{ba}{}^{x}.$$

Thus we have

LEMMA 1. *The submanifold $\overline{M}$ is minimal if and only if so is the submanifold M.*

Moreover, transvecting $A_{\gamma}{}^{\alpha}{}_{y}$ to (3.30) and using (3.21), (3.24) and (3.29) imply

$$A_{\beta\alpha}{}^{x}A_{\gamma}{}^{\alpha}{}_{y}=(A_{ba}{}^{x}A_{c}{}^{a}{}_{y}+\phi_{b}{}^{x}\phi_{cy})E_{\beta}{}^{b}E_{\gamma}{}^{c}+A_{ba}{}^{x}\phi_{y}{}^{a}E_{\beta}{}^{b}\xi_{\gamma}+A_{b}{}^{a}{}_{y}\phi_{a}{}^{x}\xi_{\beta}E_{\gamma}{}^{b}+(\phi_{\alpha}{}^{x}\phi_{y}{}^{\alpha})\xi_{\beta}\xi_{\alpha}, \tag{3.31}$$

which and (2.20) give

$$A_{\beta\alpha}{}^{x}A_{\gamma}{}^{\alpha}{}_{y}-A_{\gamma\alpha}{}^{x}A_{\beta}{}^{\alpha}{}_{y}=(\phi_{d}{}^{x}\phi_{cy}-\phi_{c}{}^{x}\phi_{dy}+A_{de}{}^{x}A_{c}{}^{e}{}_{y}-A_{ce}{}^{x}A_{d}{}^{e}{}_{y})E_{\beta}{}^{d}E_{\gamma}{}^{c} \tag{3.32}$$
$$+(\nabla_{d}\phi_{y}{}^{x})(B_{\beta}{}^{d}\xi_{\gamma}-E_{\gamma}{}^{d}\xi_{\beta}),$$

that is,

$$K_{\beta\gamma y}{}^{x}=K_{dcy}{}^{x}E_{\beta}{}^{d}E_{\gamma}{}^{c}+2\phi_{dc}\phi_{y}{}^{x}E_{\beta}{}^{d}E_{\gamma}{}^{c}+(\nabla_{d}\phi_{y}{}^{x})(E_{\beta}{}^{d}\xi_{\gamma}-E_{\gamma}{}^{d}\xi_{\beta}). \tag{3.33}$$

which are the relations between the connections in the normal bundles of M in CP^{m} and of $\overline{M}$ in S^{2m+1}.

LEMMA 2. *In order that the connection in the normal bundle of $\overline{M}$ in S^{2m+1} is flat, it is necessary and sufficient that the curvature tensor $K_{dcy}{}^{x}$ of the connection in the normal bundle of M on CP^{m} is expressed by*

$$K_{dcy}{}^{x}=-2\phi_{dc}\phi_{y}{}^{x}$$

and $\nabla_{d}\phi_{y}{}^{x}=0$.

Finally we apply the operator $\nabla_{c}=E_{c}{}^{\gamma}\overline{\nabla}_{\gamma}$ to (3.30). Then we have

$$E_{c}{}^{\gamma}\overline{\nabla}_{\gamma}A_{\beta\alpha}{}^{x}=(\nabla_{c}A_{ba}{}^{x})E_{\beta}{}^{b}E_{\alpha}{}^{a}+A_{ba}{}^{x}E_{c}{}^{\gamma}(\overline{\nabla}_{\gamma}E_{\beta}{}^{b})E_{\alpha}{}^{a}+A_{ba}{}^{x}E^{b}E_{c}{}^{\gamma}\overline{\nabla}_{\gamma}E_{\alpha}{}^{a}$$

$$+E_c^{\ \gamma}(\overline{\nabla}_\gamma\xi_\beta)\phi_\alpha^{\ x}+\xi_\beta E_c^{\ \gamma}\overline{\nabla}_\gamma\phi_\alpha^{\ x}+E_c^{\ \gamma}(\overline{\nabla}_\gamma\phi_\beta^{\ x})+\phi_\beta^{\ x}E_c^{\ \gamma}\overline{\nabla}_\gamma\xi_\alpha,$$

from which, substituting (3.10) with $h_b^{\ a}=-\phi_b^{\ a}$, (3.25) and (3.26),

$$E_c^{\ \gamma}\overline{\nabla}_\gamma A_{\beta\alpha}^{\ \ x}=(\nabla_c A_{ba}^{\ \ x})E_\beta^{\ b}E_\alpha^{\ a}-A_{ba}^{\ \ x}\phi_c^{\ b}(\xi_\beta E_\alpha^{\ a}+\xi_\alpha E_\beta^{\ a})+\phi_{\gamma\beta}E_c^{\ \gamma}\phi_\alpha^{\ x}$$
$$+\phi_{\gamma\alpha}E_c^{\ \gamma}\phi_\beta^{\ x}+\xi_\beta E_c^{\ \gamma}(A_{\gamma\alpha}^{\ \ y}\phi_y^{\ x}-A_{\gamma\delta}^{\ \ x}\phi_\alpha^{\ \delta})+\xi_\alpha E_c^{\ \gamma}(A_{\gamma\beta}^{\ \ y}\phi_y^{\ x}-A_{\gamma\delta}^{\ \ x}\phi_\beta^{\ \delta}),$$

or using (3.21) and (3.29),

$$E_c^{\ \gamma}\overline{\nabla}_\gamma A_{\beta\alpha}^{\ \ x}=(\nabla_c A_{ba}^{\ \ x}+\phi_{cb}\phi_a^{\ x}+\phi_{ca}\phi_b^{\ x})E_\beta^{\ b}E_\alpha^{\ a}-(A_{ba}^{\ \ x}\phi_c^{\ b}+A_{bc}^{\ \ x}\phi_a^{\ b}-A_{ca}^{\ \ y}\phi_y^{\ x})(\xi_\beta E_\alpha^{\ a}$$
$$+E_\beta^{\ a}\xi_\alpha)+2(\phi_c^{\ y}\phi_y^{\ x})\xi_\beta\xi_\alpha.$$

Transvecting the above equation with $E_\delta^{\ c}$ and changing the index δ with γ, we can easily find

$$\overline{\nabla}_\gamma A_{\beta\alpha}^{\ \ x}=(\nabla_c A_{ba}^{\ \ x}+\phi_{cb}\phi_a^{\ x}+\phi_{ca}\phi_b^{\ x})E_\gamma^{\ c}E_\beta^{\ b}E_\alpha^{\ a}-(A_{ba}^{\ \ x}\phi_c^{\ b}+A_{bc}^{\ \ x}\phi_a^{\ b}$$
$$-A_{ca}^{\ \ y}\phi_y^{\ x})(E_\gamma^{\ c}\xi_\beta E_\alpha^{\ a}+E_\gamma^{\ c}E_\beta^{\ a}\xi_\alpha+\xi_\gamma E_\beta^{\ a}E_\alpha^{\ c})+2(\phi_c^{\ y}\phi_y^{\ x})(E_\gamma^{\ c}\xi_\beta\xi_\alpha$$
$$+\xi_\gamma E_\beta^{\ c}\xi_\alpha+\xi_\gamma\xi_\beta E_\alpha^{\ c}). \tag{3.34}$$

Thus we have

LEMMA 3. *In order that the second fundamental tensor of $\overline{M}$ in S^{2m+1} is parallel it is necessary and sufficient that the following equations are valid on M:*

$$\nabla_c A_{ba}^{\ \ x}+\phi_{cb}\phi_a^{\ x}+\phi_{ca}\phi_b^{\ x}=0, \tag{3.35}$$

$$A_{ba}^{\ \ x}\phi_c^{\ b}+A_{bc}^{\ \ x}\phi_a^{\ b}-A_{ca}^{\ \ y}\phi_y^{\ x}=0 \tag{3.36}$$

and

$$\phi_c^{\ y}\phi_y^{\ x}=0 \ (\textit{equivalently}\ \phi_c^{\ b}\phi_a^{\ y}=0). \tag{3.37}$$

4. Anti-holomorphic submanifold of CP^m

A complex or holomorphic submanifold of a complex manifold is defined by the fact that at any point of the submanifold M the tangent space is invariant under the action of the almost complex structure $\phi_j^{\ i}$ of the ambient manifold, that is, for any $P\in M$, $T_p(M)=\phi(T_p(M))$. Since $\phi_j^{\ i}\phi_i^{\ h}=-\delta_j^{\ h}$, this condition is equivalent to the fact that, at any point of M, the normal space $N_p(M)$ is invariant under $\phi_j^{\ i}$, that is, $\phi(N_p(M))=N_p(M)$. In this point of view Okumura

[5] considered such a submanifold of a complex manifold that at any point of the submanifold

$$N_p(M)\perp\phi(N_p(M)),$$

and called this submanifold an anti-holomorphic (generic) submanifold.

From now we consider anti-holomorphic submanifolds of a complex projective space CP^m. According to our notation a submanifold of CP^m is anti-holomorphic if and only if

$$\phi_y^{\ x}=0 \tag{4.1}$$

at each point of the submanifold.

First of all we have

LEMMA 4. *Let M be anti-holomorphic submanifold of CP^m. The connection in the normal bundle of CP^m is flat if and only if the connection in the normal bundle of M in S^{2m+1} is flat.*

PROOF. It follows immediatley from lemma 2 and (4.1).

LEMMA 5. *On an n-dimensional anti-holomorphic submanifold M of $CP^{(n+p)/2}$ the following inequality.*

$$\|\nabla_c A_{ba}^{\ \ x}\|^2\geq 2p(n-p)$$

holds and that equality holds if and only if

$$\nabla_c A_{ba}^{\ \ x}+\phi_{ca}\phi_b^{\ x}+\phi_{cb}\phi_a^{\ x}=0,$$

where $\|\nabla_c A_{ba}^{\ \ x}\|^2=(\nabla_c A_b^{\ a}{}_x)(\nabla_d A_a^{\ b}{}_y)g^{cd}g^{xy}$. Moreover, when the connection in the normal bundle of M is flat, the equality implies

$$A_{ba}^{\ \ x}\phi_c^{\ a}+A_{ca}^{\ \ x}\phi_b^{\ a}=0.$$

PROOF. Putting

$$\overset{*}{\nabla}_c A_{ba}^{\ \ x}=\nabla_c A_{ba}^{\ \ x}+\phi_{ca}\phi_b^{\ x}+\phi_{cb}\phi_a^{\ x}$$

and using the equation (2.23) of Codazzi with $c=4$, we can easily verify that

$$\|\overset{*}{\nabla}_c A_{ba}^{\ \ x}\|^2=\|\nabla_c A_{ba}^{\ \ x}\|^2-(\phi_{cb}\phi^{cb})(\phi_b^{\ x}\phi_x^{\ b})$$

with the help of $\phi_b^{\ a}\phi_a^{\ x}=0$, which implies our first assertion.

Next, we assume that the connection in the normal bundle of M is flat, that is $K_{bay}^{\ \ \ x}=0$. Then the equation (2.24) with $\phi_y^{\ x}=0$ yields

(4.2) $$\phi_b^{\ x}\phi_{ay}-\phi_a^{\ x}\phi_{by}+A_{be}^{\ \ x}A_{a\ y}^{\ e}-A_{ae}^{\ \ x}A_{b\ y}^{\ e}=0.$$

If $\|\nabla_c A_{ba}^{\ \ x}\|^2=2p(n-p)$, we have

(4.3) $$\nabla_c A_{ba}^{\ \ x}=-\phi_{ca}\phi_b^{\ x}-\phi_{cb}\phi_a^{\ x}.$$

Now, differentiating (4.2) covariantly and using (2.19) with $\phi_y^{\ x}=0$ and (4.3), we can see that

$$\begin{aligned}&-A_{ce}^{\ \ x}\phi_b^{\ e}\phi_{ay}-\phi_b^{\ x}A_{c\ y}^{\ e}\phi_{ae}+A_{ce}^{\ \ x}\phi_a^{\ e}\phi_{by}+\phi_a^{\ x}A_{c\ y}^{\ e}\phi_{be}\\&-(\phi_{cb}\phi_e^{\ x}+\phi_{ce}\phi_b^{\ x})A_{a\ y}^{\ e}-A_{be}^{\ \ x}(\phi_{ca}\phi_y^{\ e}+\phi_c^{\ e}\phi_{ay})\\&+(\phi_{ca}\phi_e^{\ x}+\phi_{ce}\phi_a^{\ x})A_{b\ y}^{\ e}+A_{ae}^{\ \ x}(\phi_{cb}\phi_y^{\ e}+\phi_c^{\ e}\phi_{by})=0,\end{aligned}$$

from which, transvecting ϕ^{ay} and using (2.15)—(2.17) with $\phi_y^{\ x}=0$, we obtain

(4.4) $$(p-1)(A_{ce}^{\ \ x}\phi_b^{\ e}+A_{be}^{\ \ x}\phi_c^{\ e})-A_{ae}^{\ \ x}\phi_c^{\ e}\phi^{ay}+A_{a\ y}^{\ e}\phi_{ce}\phi_b^{\ x}\phi^{ay}=0.$$

Taking the skew-symmetric part of the above equation, we get

$$A_{a\ y}^{\ e}\phi^{ay}(\phi_{ce}\phi_b^{\ x}-\phi_{be}\phi_c^{\ x})+A_{ae}^{\ \ x}\phi_b^{\ e}\phi_{cy}\phi^{ay}-A_{ae}^{\ \ x}\phi_c^{\ e}\phi_{by}\phi^{ay}=0,$$

from which, transvecting with $\phi_x^{\ b}$,

$$(p-1)A_{ae}^{\ \ x}\phi_c^{\ e}\phi_x^{\ a}=0,$$

which and the last equation imply

$$A_{ae}^{\ \ x}\phi_c^{\ e}\phi_{by}\phi^{ay}-A_{ae}^{\ \ x}\phi_b^{\ e}\phi_{cy}\phi^{ay}=0,$$

provided $p>1$. Transvecting this equation with $\phi_z^{\ b}$ gives

$$A_{ae}^{\ \ x}\phi_c^{\ e}\phi_z^{\ a}=0,$$

which and (4.4), provided $p>1$, yield

$$A_{ce}^{\ \ x}\phi_b^{\ e}+A_{be}^{\ \ x}\phi_c^{\ e}=0.$$

But fortunately when $p=1$, that is, when the submanifold is real hypersurface, Maeda [3] already proved this implication. Hence we complete the proof of lemma 4.

As a converse of lemma 4, we prove

LEMMA 6. *Let M be an n-dimensional anti-holomorphic, minimal submanifold of* $CP^{(n+p)/2}$. *If the connection in the normal bundle of M is flat, and if*

$$A_{ce}^{\ \ x}\phi_b^{\ e}+A_{be}^{\ \ x}\phi_c^{\ e}=0$$

at every point of M, then

$$\overline{\nabla}_\gamma A_{\beta\alpha}{}^x=0$$

and

$$A_{\beta\alpha}{}^x A^{\beta\alpha}{}_x=(n+1)p.$$

PROOF. We assume that

$$A_{ba}{}^x\phi_c{}^b+A_{bc}{}^x\phi_a{}^b=0 \tag{4.5}$$

at every point of M. Transvecting (4.5) with $\phi_y{}^c$ and using (2.16) with $\phi_y{}^x=0$, we have

$$A_{bc}{}^x\phi_y{}^c\phi_a{}^b=0,$$

from which transvecting with $\phi_d{}^a$,

$$A_{dc}{}^x\phi_y{}^c=P_{yz}{}^x\phi_d{}^z, \tag{4.6}$$

where we have put $P_{yz}{}^x=A_{bc}{}^x\phi_y{}^b\phi_z{}^c$. Applying the operator ∇_b to the both sides of (4.6) and then taking the skew-symmetric part with respect to the indices b and d, we obtain

$$(\nabla_b A_{dc}{}^x-\nabla_d A_{bc}{}^x)\phi_y{}^c+A_{dc}{}^x A_b{}^e{}_y\phi_e{}^c-A_{bc}{}^x A_d{}^e{}_y\phi_e{}^c$$
$$=(\nabla_b P_{yz}{}^x)\phi_d{}^z-(\nabla_d P_{yz}{}^x)\phi_b{}^z-P_{yz}{}^x A_{be}{}^z\phi_d{}^e+P_{yz}{}^x A_{de}{}^z\phi_b{}^e$$

with the help of (2.19) with $\phi_y{}^x=0$. We substitute (2.23) with $c=4$ in the last equation and use that $K_{dcy}{}^x=0$, that is,

$$A_{de}{}^x A_c{}^e{}_y-A_{ce}{}^x A_d{}^e{}_y=-\phi_d{}^x\phi_{cy}+\phi_c{}^x\phi_{dy}.$$

Then we can see that

$$\begin{aligned}&-2\phi_{bd}\delta_y^x+2A_{ec}{}^x A_d{}^c{}_y\phi_b{}^e\\&=(\nabla_b P_{yz}{}^x)\phi_d{}^z-(\nabla_d P_{yz}{}^x)\phi_b{}^z-2P_{yz}{}^x A_{be}{}^z\phi_d{}^e.\end{aligned} \tag{4.7}$$

Transvecting (4.7) with $\phi_w{}^b$and using (2.16) witg $\phi_y{}^z=0$ and (4.5), we find

$$\nabla_d P_{yw}{}^x=\phi_w{}^b(\nabla_b P_{yz}{}^x)\phi_d{}^z,$$

which and $P_{yz}{}^x=P_{zy}{}^x$imply

$$(\nabla_b P_{yz}{}^x)\phi_d{}^z=\phi_y{}^e(\nabla_e P_{zw}{}^x)\phi_b{}^w\phi_d{}^z.$$

Therefore (4.7) reduces to

$$-\phi_{bd}\delta^x_y + A_{ec}{}^x A_d{}^c{}_y \phi_b{}^e = P_{yz}{}^x A_{de}{}^z \phi_b{}^e,$$

from which, transvecting with $\phi_a{}^b$, we have

$$(4.8)\qquad (g_{ad} - \phi_a{}^z \phi_{dz})\delta^z_y - A_{ae}{}^x A_d{}^e{}_y + A_{ec}{}^x A_d{}^c{}_y \phi_a{}^z \phi_z{}^e$$
$$= -P_{yz}{}^x A_{da}{}^z + P_{yz}{}^x A_{de}{}^z \phi_a{}^w \phi_w{}^e.$$

On the other hand, by using the definition of $P_{yz}{}^x$ and taking account of $K_{dcy}{}^x = 0$ and (2.20) with $\phi_y{}^x = 0$, we can see that

$$P_{yz}{}^x A_{de}{}^z \phi_a{}^w \phi_w{}^e = A_{be}{}^x A_d{}^b{}_y \phi_w{}^e \phi_a{}^w - \phi_d{}^z \phi_{az} \delta^x_y + \phi_a{}^x \phi_{dy}$$

and consequently from (4.8)

$$(4.9)\qquad A_{ca}{}^x A_b{}^a{}_y = P_{yz}{}^x A_{cb}{}^z + g_{cb}\delta^x_y - \phi_c{}^x \phi_{by}.$$

Therefore substituting (4.9) into (3.31) gives

$$(4.10)\qquad A_{\beta\alpha}{}^x A_\gamma{}^\alpha{}_y = P_{yz}{}^x A_{\beta\gamma}{}^z + g_{\beta\gamma}\delta^x_y$$

and hence

$$A_{\beta\alpha}{}^x A^{\beta\alpha}{}_x = (n+1)p.$$

We now compute the Laplacian ΔF of the function $F = A_{\beta\alpha}{}^x A^{\beta\alpha}{}_x$, which is globally defined on $\overline{M}$, where $\Delta = g^{\beta\alpha}\overline{\nabla}_\beta \overline{\nabla}_\alpha$. We then have

$$\frac{1}{2}\Delta F = g^{\gamma\delta}(\overline{\nabla}_\gamma \overline{\nabla}_\delta A_{\beta\alpha}{}^x) A_{\beta\alpha}{}^x + \|\overline{\nabla}_\gamma A_{\beta\alpha}{}^x\|^2.$$

On the other hand, since from our assumption and lemmas 1 and 3

$$K_{\beta\alpha y}{}^x = 0,\quad g^{\beta\alpha} A_{\beta\alpha}{}^x = 0,$$

using the Ricci identity and the equation (2.23) of Codazzi with $c=4$, we can easily obtain

$$\frac{1}{2}\Delta F = K_\gamma{}^\alpha A_{\beta\alpha}{}^x A^{\gamma\beta}{}_x - K_{\gamma\delta\beta\alpha} A^{\gamma\alpha}{}_x A^{\delta\beta x} A + \|\overline{\nabla}_\gamma A_{\beta\alpha}{}^x\|^2,$$

where $K_\gamma{}^\alpha$ is the Ricci tensor of $\overline{M}$ given by

$$K_\gamma{}^\alpha = n\delta^\alpha_\gamma - A_\gamma{}^\beta{}_x A_\beta{}^{\alpha x}.$$

Therefore the expression above of $\frac{1}{2}\Delta F$ reduces to

$$\frac{1}{2}\Delta F = (n+1) A_{\beta\alpha}{}^x A^{\beta\alpha}{}_x - (A_{\beta\alpha y} A^{\beta\alpha}{}_x)(A_{\gamma\delta}{}^y A^{\gamma\delta x}) + \|\overline{\nabla}_\gamma A_{\beta\alpha}{}^x\|^2,$$

from which, taking account of $F=(n+1)p=$constant and

$$(n+1)A_{\beta\alpha}{}^{x}A^{\beta\alpha}{}_{x}=(A_{\beta\alpha y}A^{\beta\alpha}{}_{x})(A_{\gamma\delta}{}^{y}A^{\gamma\delta x}),$$

we have

$$\|\overline{\nabla}_{\gamma}A_{\beta\alpha}{}^{x}\|^{2}=0,$$

which gives

$$\overline{\nabla}_{\gamma}A_{\beta\alpha}{}^{x}=0$$

Thus we complete the proof of lemma 6.

Combinning lemma 3, lemma 5 and lemma 6, we have

PROPOSITION 7. *Let M be an n-dimensional anti-holomorphic, minimal submanifold of $CP^{(n+p)/2}$. Suppose the connection in the normal bundle of M is flat. Then the following conditions* (1)—(3) *are equivalent to each other:*

(1) $\|\nabla_{c}A_{ba}{}^{x}\|^{2}=2p(n-p)$.

(2) $A_{ce}{}^{x}\phi_{b}{}^{e}+A_{be}{}^{x}\phi_{c}{}^{e}=0$.

(3) $\overline{\nabla}_{\gamma}A_{\beta\alpha}{}^{x}=0$.

Thus, combinning proposition 7 and theorem A, we have

COROLLARY. *Let M be a complete n-dimensional anti-holomorphic minimal submanifold of CP^{m} whose normal connection is flat. If the second fundamental tensor $A_{ba}{}^{x}$ of M satisfies*

$$A_{ce}{}^{x}\phi_{b}{}^{e}+A_{be}{}^{x}\phi_{c}{}^{e}=0,$$

then M is

$$\tilde{\pi}(S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k)),\quad r_t=\sqrt{m_t/(n+1)}\ (t=1,\cdots,k),\quad n+1=\sum_{i=1}^{k} m_i$$

where $m_1,\cdots,m_k$ are odd numbers such that $m_1,\cdots,m_k\geq 1$, $p=k-1$.

From now we consider a vector field whose components are given by

$$\phi_{x}{}^{e}\nabla_{e}\phi^{xb}-(\nabla_{e}\phi_{x}{}^{e})\phi^{xb}.$$

Then we can easily find

$$\nabla_{b}(\phi_{x}{}^{e}\nabla_{e}\phi^{xb})-\nabla_{b}(\phi^{xb}\nabla_{e}\phi_{x}{}^{e})=(\nabla_{b}\phi_{x}{}^{e})(\nabla_{e}\phi^{xb})-\|\nabla_{b}\phi_{x}{}^{b}\|^{2}$$
$$+K_{ba}\phi_{x}{}^{b}\phi^{xa}-K_{baxy}\phi^{ax}\phi^{by}$$

$$=K_{ba}\phi_x{}^b\phi^{xa}-K_{baxy}\phi^{ax}\phi^{by}-\|\nabla_b\phi_{ax}\|^2-\|\nabla_b\phi_x{}^b\|^2$$
$$+\frac{1}{2}\|\nabla_b\phi_{ax}+\nabla_a\phi_{bx}\|^2,$$

from which, substituting (2.19),

$$\nabla_b(\phi_x{}^e\nabla_e\phi^{xb})-\nabla_b(\phi^{xb}\nabla_e\phi_x{}^e)=p(n-1)-A_{ba}{}^xA^{ba}{}_x+A_xA_{cb}{}^x\phi_y{}^c\phi^{by}$$
$$-K_{baxy}\phi^{ax}\phi^{by}+\frac{1}{2}\|A_{bex}\phi_a{}^e+A_{aex}\phi_b{}^e\|^2$$

where we have assumed the submanifold is anti-holomorphic.

Thus we have

PROPOSITION 8. *Let M be an n-dimensional complex projective space $CP^{(n+p)/2}$. Then the condition*

$$\int_M\{p(n-1)-A_{ba}{}^xA^{ba}{}_x+A_xA_{ba}{}^x\phi_y{}^b\phi^{ay}-K_{baxy}\phi^{ax}\phi^{by}\}*1\geq 0$$

is equivalent to the condition (2) *stated in* proposition 7.

Combinning proposition 7, proposition 8 and theorem A, we have

COROLLARY. *Let M be an n-dimensional compact, minimal, anti-holomorphic submanifold of $CP^{(n+p)/2}$ whose normal connection is flat. If the second fundamental $A_{ba}{}^x$ of M satisfies*

$$A_{ba}{}^xA^{ba}{}_x\leq p(n-1)$$

at each point of M, then M is

$$\tilde{\pi}(S^{m_1}(r_1)\times\cdots\cdots\times S^{m_k}(r_k)),\ r_t=\sqrt{m_t/(n+1)}\ (t=1,\cdots,k)\ n+1=\sum_{i=1}^{k}m_i$$

where $m_1,\cdots\cdots,m_k$ are odd numbers such that $m_1,\cdots,m_k\geq 1$, $p=k-1$.

As a special occurence, we consider the case $p=1$. Then we have

COROLLARY(Lawson [1]). *Let M be a compact, real minimal hypersurface of $CP^{(n+1)/2}$ on which the inequality*

$$A_{ba}A^{ba}\leq n-1$$

holds. Then M is

$$\tilde{\pi}\left(S^k\left(\sqrt{\frac{k}{n+1}}\right)\times S^{n-k}\left(\sqrt{\frac{n-k+1}{n+1}}\right)\right),\ 1\leq k\leq n,$$

where k is an odd number.

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