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(ϕ, ξ, η)-CONNECTIONS IN ALMOST PARACONTACT MANIFOLD

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1. Introduction

Let M be an n-dimensional differentiable manifold equipped with a triplet (ϕ, ξ, η), where ϕ is a tensor field of type (1.1), ξ is a vector field and η a 1-form defined on M satisfying:

(1.1) $\phi^2 = I - \eta\otimes\xi$, $\phi(\xi)=0$, $\eta(\xi)=1$ and $\eta\circ\phi=0$.

Then the manifold is said to have an *almost paracontact structure* (ϕ, ξ, η) and M is called an *almost paracontact manifold* [2].

The almost paracontact structure (ϕ, ξ, η) on an almost paracontact manifold is said to be *normal* if the tensor,

$$N^i_{jk} = \phi^1_j(\partial_1\phi^i_k - \partial_k\phi^i_1) - \phi^1_k(\partial_1\phi^i_j - \partial_j\phi^i_1) - \eta_j\partial_k\xi^i + \eta_k\partial_j\xi^i \tag{1.2}$$

vanishes. There corresponds three tensors N^i_j, N_{jk} and N_j, to the tensor defined by (1.2) and are given by [2],

$$\begin{aligned} N_{jk} &= \phi^1_j(\partial_1\eta_k - \partial_k\eta_1) - \phi^1_k(\partial_1\eta_j - \partial_j\eta_1), \\ N^i_j &= \mathcal{L}_\xi\phi^i_j \text{ and } N_j = \mathcal{L}_\xi\eta_j, \end{aligned} \tag{1.3}$$

where $\mathcal{L}_\xi$ means the Lie-derivative w.r.t. ξ. It is known that if N^i_{jk} vanishes then N^j_k, N_{ij} and N_j vanish.

In our previous paper [4], we have shown that, we can give a natural 3-π-structure on an almost paracontact manifold whose fundamental tensor is given by

$$F^i_j = \frac{\lambda}{2}(-\delta^i_j + 3\xi^i\eta_j + w_1(w_1-1)\phi^i_j) \tag{1.4}$$

where w_1 is cuberoot of unity $(w_1 \neq 1)$ and λ is any nonzero constant.

The torsion tensor of this 3-π-structure is given by

$$\begin{aligned} T^i_{jk} = \frac{1}{4}\{ &-N^i_{jk} - 3\xi^i(\partial_k\eta_j - \partial_j\eta_k) + 5\xi^i(N_j\eta_k - \eta_j N_k) \\ &+ \xi^i\phi^p_j\phi^q_k(\partial_q\eta_p - \partial_p\eta_q) + N^i_p(\eta_k\phi^p_j - \eta_j\phi^p_k)\}. \end{aligned} \tag{1.5}$$

Further we say that 3-π-structure on an almost paracontact manifold is integrable if $T^i_{jk}=0$.

Let ${}^2F^i_k = F^i_j F^j_k$, where F^i_j is given by (1.4) then

$$(1.6) \qquad {}^2F^i_k = \frac{\lambda^2}{2}(-\delta^i_j + 3\xi^i\eta_j - w_1(w_1-1)\phi^i_k).$$

In the present paper, we study the existence and properties of a connection, on an almost paracontact manifold which leaves all the three structure tensors ϕ, ξ and η, covariant constant, by using the theory of r-π-structures [1], and call such a connection as (ϕ, ξ, η)*-connection.*

2. (ϕ, ξ, η)-Connections

C. J. Hsu [1] has introduced the concept of a π-connection, which is a connection that leaves the fundamental tensor of r-π-structure covariant constant. He has shown that if Γ^i_{jk} is any linear connection of the manifold and let $\grave{\Gamma}^i_{jk}$ be a connection defined by

$$(2.1) \qquad \grave{\Gamma}^i_{jk} = \Gamma^i_{jk} + M^i_{jk},$$

with

$$(2.2) \qquad M^i_{jk} = \frac{1}{3\lambda^3}\{(\nabla_k F^1_j)\, {}^2F^i_1 + (\nabla_k {}^2F^1_j)\, F^i_1\},$$

where ∇ denote the covariant differentiation with respect to Γ^i_{jk}, then $\grave{\Gamma}^i_{jk}$ is a π-connection of the differentiable manifold with 3-π-structure whose fundamental tensor is given by (1.4).

Since $\grave{\Gamma}^i_{jk}$ leaves F^i_j covariant constant, it also leaves ${}^2F^i_j$ covariant constant. From (1.4) and (1.6), we observe that $\xi^i\eta_j$ and ϕ^i_j can be expressed as a linear combination of F^i_j and ${}^2F^i_j$, this shows that $\grave{\Gamma}^i_{jk}$ also leaves $\xi^i\eta_j$ and ϕ^i_j covariant constant. If we substitute the values of F^i_j and ${}^2F^i_j$ in (2.2) we get

$$(2.3) \qquad M^i_{jk} = \frac{1}{2}\{2\xi^i(\nabla_k\eta_j) - \eta_j(\nabla_k\xi^i) + 3\xi^i\eta_j\eta_1(\nabla_k\xi^l) + \phi^i_1(\nabla_k\phi^1_j)\}.$$

Using (2.1) and (2.3), we get by a straight forward calculation

$$(2.4) \qquad \grave{\nabla}_k\xi^i = \xi^i\eta_1(\nabla_k\xi^l)$$

where $\grave{\nabla}$ denote the covariant differentiation w.r.t. π-connection $\grave{\Gamma}^i_{jk}$

THEOREM 2.1. *On an almost paracontact manifold M there exists a (ϕ,ξ,η)-connection.*

PROOF. It is known that, there exists a symmetric connection on an almost paracontact manifold which leaves covariant constant [3]. If we take Γ^i_{jk} in (2.1) to be such a connection then it follows from (2.4) that

$$\grave{\nabla}_k \xi^i = 0. \tag{2.5}$$

Since a π-connection already leaves ϕ^i_j covariant constant we have by (theorem 2.3 [3]),

$$\grave{\nabla}_k \xi^i = \lambda_k \xi^i \text{ and } \grave{\nabla}_k \eta_j = -\lambda_k \eta_j \tag{2.6}$$

where λ_k is a covariant vector on M.

Using (2.5) and (2.6) we get $\lambda_k=0$ and there by $\grave{\nabla}_k \eta_j=0$, proving that $\grave{\Gamma}^i_{jk}$ is a (ϕ,ξ,η)-connection on M.

THEOREM 2.2. *A connection ${}^+\Gamma^i_{jk}$ on an almost paracontact manifold is a (ϕ,ξ,η)-connection if and only if can be expressed as*

$${}^+\Gamma^i_{jk} = {}^*\Gamma^i_{jk} + U^i_{jk}, \tag{2.7}$$

where

$$U^i_{jk} = \frac{1}{3}\left\{\sigma^i_{jk} + \frac{1}{\lambda^3}({}^2F^s_j\, \sigma^r_{sk} F^i_r + F^s_j\, \sigma^r_{sk}\, {}^2F^i_r\right\}, \tag{2.8}$$

with some tensor σ^i_{jk} satisfying

$$\xi^j \sigma^i_{jk} \eta_i = 0$$

and ${}^*\Gamma^i_{jk}$ is a linear connection which leaves ξ^i covariant constant.

PROOF. It is known that [1] a linear connection ${}^+\Gamma^i_{jk}$ is a π-connection if and only if it can be expressed as (2.7) with some tensor σ^i_{jk}.

Let $\overset{+}{\nabla}_k \xi^i$ and $\overset{*}{\nabla}_k \xi^i$ be respectively the convariant derivatives of ξ^i w.r.t. ${}^+\Gamma^i_{jk}$ and ${}^*\Gamma^i_{jk}$ then

$$\overset{+}{\nabla}_k \xi^i = \overset{*}{\nabla}_k \xi^i + U^i_{jk} \xi^j = U^i_{jk} \xi^j.$$

Thus, as in the proof of theorem (2.1), the condition for ${}^+\Gamma^i_{jk}$, to be a (ϕ,ξ,η)-connection is

$$U^i_{jk} \xi^j = 0. \tag{2.9}$$

Substituting (2.8), (1.4) and (1.6) in (2.9), we get after some calculations the following condition

$$\xi^i \sigma^i_{jk} \eta_i = 0.$$

Thus the theorem is proved.

THEOREM 2.3. *Let M be an almost paracontact manifold with almost paracontact structure (ϕ, ξ, η). If $N_j=0$ then there exists a (ϕ, ξ, η)-connection whose torsion tensor is equal to the tensor T^i_{jk} in* (1.5).

PROOF. Let $\overset{*}{\Gamma}{}^i_{jk}$ be the induced π-connection by a symmetric connection as in theorem (2.1), then it follows from the theory of π-structures [1], that the connection defined by

$$\hat{\Gamma}^i_{jk} = \overset{*}{\Gamma}{}^i_{jk} - \frac{2}{3}\left\{ {}^*S^i_{jk} + \frac{1}{\lambda^3} ({}^2F^s_j \, {}^*S^r_{sk} F^i_r + F^s_j \, {}^*S^r_{sk} \, {}^2F^i_r) \right\} \tag{2.10}$$

is a distinguished π-connection, that is the π-connection having the torsion tensor as T^i_{jk} in (1.5). In (2.10) ${}^*S^i_{jk}$ is the torsion tensor of the connection ${}^*\Gamma^i_{jk}$.

On the other hand since ${}^*\Gamma^i_{jk}$ is a (ϕ, ξ, η)-connection we see from theorm (2.2) that $\hat{\Gamma}^i_{jk}$ in (2.10) is also a (ϕ, ξ, η)-connection if and only if, the following condition is satisfied.

$$\xi^j_{\eta 1} \, {}^*S^i_{jk} = 0. \tag{2.11}$$

Now let us calculate ${}^*S^i_{jk}$ for this, from theorem (2.1) we have

$${}^*\Gamma^i_{jk} = \Gamma^i_{jk} + M^i_{jk}$$

where Γ^i_{jk} is symmetric affine connection which leaves ξ^i covariant constant and

$$M^i_{jk} = \frac{1}{2}\left\{ 2\xi^i (\nabla_k \eta_j) + \phi^i_1 (\nabla_k \phi^1_j) \right\}$$

where ∇ denote the covariant differentiation w.r.t. Γ^i_{jk}.

Thus

$${}^*S^i_{jk} = \frac{1}{2}(M^i_{jk} - M^i_{kj}) = \frac{1}{2}\{2\xi^i (\nabla_k \eta_j) + \phi^i_1 (\nabla_k \phi^1_j) - 2\xi^i (\nabla_j \eta_k) - \phi^i_1 (\nabla_j \phi^1_k)\}.$$

In view of above equation, (2.11) is just nothing but

$$N_k=0,$$

which is given. Hence, the theorem is proved.

THEOREM 2.4. *Let M be an almost paracontact manifold with*

(i) η *is closed,*

(ii) $N^i_j=0.$

Then there exists a (ϕ, ξ, η)-*connection whose torsion tensor is equal to* $-\frac{1}{4} N^i_{jk}$.

PROOF. Since η is closed we have

$$\partial_j \eta_k - \partial_k \eta_j = 0.$$

Transvecting above equation by ξ^j, we get

$$N_k=0.$$

Hence, if η is closed and $N^i_j=0$, we get from (1.5)

$$T^i_{jk} = -\frac{1}{4} N^i_{jk}.$$

Using theorem (2.3), we get the result.

3. Symmetric (ϕ, ξ, η)-connection

In this section we establish the existance of a symmetric (ϕ, ξ, η)-connection on an almost paracontact manifold.

THEOREM 3.1. *Let M be an almost paracontact manifold with* 1-*form* η *closed. Then there exists a symmetric* (ϕ, ξ, η)-*connection on M if and only if* 3-π-*structure defined on M is integrable.*

PROOF. Let there exist a symmetric (ϕ, ξ, η)-connection on *M*. If we denote the covariant differentiation w.r.t. this connection by ∇, then obviously (1.2) and (1.3) hold with ∂ replaced by ∇. But since the connection is (ϕ, ξ, η)-connection we have

$$N^i_{jk}=0, \quad \text{consequently } N^i_j = N_{jk} = N_k = 0.$$

Hence, by (1.5), as η is closed we get $T^i_{jk}=0$ that is the 3-π-structure is integrable.

The converse follows from theorem (2.3).

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