Kyungpook Math. J.
Volume 20, Number 1
June, 1980

ON FINSLER SPACE OF RECURRENT CURVATURE TENSORS

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0. Summary

The Riemannian space of recurrent curvature was defined and studied by Ruse [8] and Walker [10]. In 1963, Móor [4] generalised this idea for Finsler spaces and defined and studied Finsler spaces of recurrent curvature. These spaces for various curvature tensors have subsequently been studied by Mishra and Pande [1], Sen [9] and Misra [3] etc. The purpose of the present paper is to study Finsler space based on the recurrency of the curvature tensors derived from non-linear connections.

1. Introduction

Let $X^i(x^k)$ and $Y_j(x^k)$ be two differentiable vector fields in a Finsler space F_n with metric tensor $g_{ij}(x, X)$ and non-linear connections $\overset{1}{\Gamma}{}^i_k(x, X)$ and $\overset{2}{\Gamma}_{jk}(x, Y)$, positively homogeneous of first degree in X and Y respectively, then we have Rund [6]:

$$\overset{1}{\Gamma}{}^i_{jk}=\Delta_j\overset{1}{\Gamma}{}^i_k\,,\quad X^j\overset{1}{\Gamma}{}^i_{jk}=\overset{1}{\Gamma}{}^i_k \tag{1.1}$$

and

$$\overset{2}{\Gamma}{}^i_{jk}=\Delta^i\overset{2}{\Gamma}_{jk},\quad Y_i\overset{2}{\Gamma}{}^i_{jk}=\overset{2}{\Gamma}_{jk}, \tag{1.2}$$

where $\Delta_j=\partial/\partial X^j$ and $\Delta^i=\partial/\partial Y_i$.

Let us suppose that if X^i undergoes a parallel displacement then so does $Y_i=g_{ij}X^j$, such that the length of a vector remains unchanged under parallel displacement, then we have [6]:

$$2G^i=\overset{1}{\Gamma}{}^i_kX^k+g^{ih}Y_j(\overset{1}{\Gamma}{}^j_{hk}X^k-\overset{1}{\Gamma}{}^j_h). \tag{1.3}$$

Assuming geodesics to be auto-parallel curves of F_n we get

$$2G^i=\overset{1}{\Gamma}{}^i_kX^k\,, \tag{1.4}$$

such that

$$(1.5)\qquad \overset{1}{\Gamma}{}^{i}_{hj}=G^{i}_{hj}+\frac{1}{2}\{\overset{1}{S}{}^{i}_{hj}-X^{k}\Delta_{j}\overset{1}{S}{}^{i}_{hk}\}$$

and

$$(1.6)\qquad \overset{2}{\Gamma}{}^{h}_{ik}=\overset{1}{\Gamma}{}^{h}_{ik}+Y_{j}\Delta^{h}\overset{1}{\Gamma}{}^{j}_{ik},$$

where

$$\overset{1}{S}{}^{i}_{hk}=2\overset{1}{\Gamma}{}^{i}_{[hk]}.$$

The covariant derivative of a tensor $T^{i}_{j}(x, X)$ is defined by [2]:

$$(1.7)\qquad T^{i}_{j,k}(x,X)=\partial_{k}T^{i}_{j}+(\Delta_{m}T^{i}_{j})(\partial_{k}X^{m})+T^{m}_{j}\overset{1}{\Gamma}{}^{i}_{mk}-T^{i}_{m}\overset{2}{\Gamma}{}^{m}_{jk}.$$

The two curvature tensors based on these coefficients of connections are given by [2]:

$$(1.8)\qquad \overset{1}{R}{}^{i}_{jkh}(x,X)=2\{\partial_{[h}\overset{1}{\Gamma}{}^{i}_{|j|k]}+(\Delta_{m}\overset{1}{\Gamma}{}^{i}_{j[k})(\partial_{k]}X^{m})+\overset{1}{\Gamma}{}^{m}_{j[k}\overset{1}{\Gamma}{}^{i}_{|m|h]}\}$$

and

$$(1.9)\qquad \overset{2}{R}{}^{i}_{jkh}(x,Y)=2\{\partial_{[h}\overset{2}{\Gamma}{}^{i}_{|j|k]}+(\Delta^{m}\overset{2}{\Gamma}{}^{i}_{j[k})(\partial_{h]}Y_{m})+\overset{2}{\Gamma}{}^{m}_{j[k}\overset{2}{\Gamma}{}^{i}_{|m|h]}\}$$

and satisfy

$$(1.10)\qquad X^{j}\overset{1}{R}{}^{i}_{jkh}=\overset{1}{R}{}^{i}_{kh},\quad Y_{j}\overset{2}{R}{}^{j}_{khl}=\overset{2}{R}_{khl},$$

such that

$$(1.11)\text{a}\qquad \overset{1}{R}{}^{i}_{jkh}(x,X)=\Delta_{j}\overset{1}{R}{}^{i}_{kh}+2(\Delta_{j}\overset{1}{\Gamma}{}^{i}_{m[k})X^{m}_{,h]}$$

and

$$(1.11)\text{b}\qquad \overset{2}{R}{}^{i}_{jkh}(x,Y)=\Delta^{i}\overset{2}{R}_{jkh}+2(\Delta^{i}\overset{2}{\Gamma}{}^{m}_{j[k})\,Y_{|m|,h]}$$

2. Recurrent curvature tensors

DEFINITION 2.1. If in a non-flat Finsler space F_n the curvature tensors $\overset{1}{R}{}^{i}_{jkh}$, $\overset{2}{R}{}^{i}_{jkh}$, $\overset{1}{R}{}^{i}_{kh}$ and $\overset{2}{R}_{jkh}$ satisfy

$$(2.1)\text{a}\qquad \overset{1}{R}{}^{i}_{jkh,l}=\lambda_{l}\overset{1}{R}{}^{i}_{jkh},$$

$$(2.1)\text{b}\qquad \overset{2}{R}{}^{i}_{jkh,l}=\lambda_{l}\overset{2}{R}{}^{i}_{jkh},$$

$$(2.1)\text{c}\qquad \overset{1}{R}{}^{i}_{kh,l}=\lambda_{l}\overset{1}{R}{}^{i}_{kh}$$

and

(2.1)d $$\overset{2}{R}_{jkh,l} = \lambda_l \overset{2}{R}_{jkh},$$

respectively, for a non-null covariant vector λ_l, then they are called *R-recurrent curvature tensors of various types* respectively.

DEFINITION 2.2. If $\overset{1}{R}{}^i_{jki} \underline{\underline{def}} \overset{1}{R}_{jk}$, $\overset{2}{R}{}^i_{jki} \underline{\underline{def}} \overset{2}{R}_{jk}$, $\overset{1}{R}{}^i_{ijk} \underline{\underline{def}} \overset{1}{R'}_{jk}$, $\overset{2}{R}{}^i_{ijk} \underline{\underline{def}} \overset{2}{R'}_{jk}$, $\overset{1}{R}_{jkhr} \underline{\underline{def}}\, g_{ir} \overset{1}{R}{}^i_{jkh}$, $\overset{2}{R}_{jkhr} \underline{\underline{def}}\, g_{ir} \overset{2}{R}{}^i_{jkh}$, $\overset{1}{R}_{khr} \underline{\underline{def}}\, g_{ir} \overset{1}{R}{}^i_{kh}$ and $\overset{2}{R}{}^i_{jk} \underline{\underline{def}}\, g^{ir} \overset{2}{R}_{jkr}$ satisfy

(2.2)a $$\overset{1}{R}{}^i_{jk,l} = \lambda_l \overset{1}{R}_{jk},$$

(2.2)b $$\overset{2}{R}_{jk,l} = \lambda_l \overset{2}{R}_{jk},$$

(2.2)c $$\overset{1}{R'}_{jk,l} = \lambda_l \overset{1}{R'}_{jk},$$

(2.2)d $$\overset{2}{R'}_{jk,l} = \lambda_l \overset{2}{R'}_{jk},$$

(2.2)e $$\overset{1}{R}_{jkhr,l} = \lambda_l \overset{1}{R}_{jkhr},$$

(2.2)f $$\overset{2}{R}_{jkhr,l} = \lambda_l \overset{2}{R}_{jkhr},$$

(2.2)g $$\overset{1}{R}_{khr,l} = \lambda_l \overset{1}{R}_{khr},$$

and

(2.2)h $$\overset{2}{R}{}^i_{jk,l} = \lambda_l \overset{2}{R}{}^i_{jk},$$

respectively, then the various curvature tensors given in the definition are called *R-recurrent curvature tensors* respectively.

Multiplying equations (2.1)a and (2.1)b by X^j and Y_i respectively we get on simplification

(2.3)a $$\overset{1}{R}{}^i_{kh,l} = \lambda_l \overset{1}{R}{}^i_{kh} - X^j_{,l} \overset{1}{R}{}^i_{jkh}$$

and

(2.3)b $$\overset{2}{R}_{jkh,l} = \lambda_l \overset{2}{R}_{jkh} - Y_{i,l} \overset{2}{R}{}^i_{jkh},$$

which by virtue of equations (2.1)c and (2.1)d imply;

THEOREM 2.1. *If* $\overset{1}{R}{}^i_{jkh}$ *and* $\overset{2}{R}{}^i_{jkh}$ *are R-recurrent curvature tensors, then the necessary and sufficient conditions for* $\overset{1}{R}{}^i_{jk}$ *and* $\overset{2}{R}_{jkh}$ *to be R-recurrent are given by* $X^j{}_{,l}\overset{1}{R}{}^i_{jkh}=0$ *and* $Y_{i,l}\overset{2}{R}{}^i_{jkh}=0$, *respectively.*

Differentiating (2.1)c and (2.1)d partially with respect to X^j and Y_i respectively and using equations (1.11)a and (1.11)b we obtain on simplification

$$\begin{aligned}(2.4)\text{a}\quad \overset{1}{R}{}^i_{jkh,l}-\lambda_l\overset{1}{R}{}^i_{jkh} &= (\Delta_j\lambda_l)\overset{1}{R}{}^i_{kh}-2\lambda_l(\Delta_j\overset{1}{\Gamma}{}^i_{m[k})X^m_{,h]}\\ &-(\Delta_m\overset{1}{R}{}^i_{kh})\overset{2}{\Gamma}{}^m_{jl}-\overset{1}{R}{}^m_{kh}\Delta_j\overset{1}{\Gamma}{}^i_{ml}+\overset{1}{R}{}^i_{mh}\Delta_j\overset{2}{\Gamma}{}^m_{kl}\\ &+\overset{1}{R}{}^i_{km}\Delta_j\overset{2}{\Gamma}{}^m_{hl}+2\{(\Delta_j\overset{1}{\Gamma}{}^i_{m[k})X^m_{,h]}\}_{,l}-(\Delta_m\overset{1}{R}{}^i_{kh})\Delta_j(\partial_lX^m)\end{aligned}$$

$$\begin{aligned}(2.4)\text{b}\quad \overset{2}{R}{}^i_{jkh,l}-\lambda_l\overset{2}{R}{}^i_{jkh} &= (\Delta^i\lambda_l)\overset{2}{R}_{jkh}-2\lambda_l(\Delta^i\overset{2}{\Gamma}{}^m_{j[k})Y_{|m|,h]}\\ &+2\{(\Delta^i\overset{2}{\Gamma}{}^m_{j[k})Y_{|m|,h]}\}_{,l}-(\Delta^m\overset{2}{R}_{jkh})\Delta^i(\partial_lY_m)\\ &+(\Delta^m\overset{2}{R}_{jkh})\overset{1}{\Gamma}{}^i_{ml}+\overset{2}{R}_{mkh}\Delta^i\overset{2}{\Gamma}{}^m_{jl}\\ &+\overset{2}{R}_{jmh}\Delta^i\overset{2}{\Gamma}{}^m_{kl}+\overset{2}{R}_{jkm}\overset{2}{\Gamma}{}^m_{hl},\end{aligned}$$

respectively. From equations (2.4)a and (2.4)b by virtue of equations (2.1)a and (2.1)b we obtain;

THEOREM 2.2. *If* $\overset{1}{R}{}^i_{kh}$ *and* $\overset{2}{R}_{jkh}$ *are R-recurrent curvature tensors then the necessary and sufficient condition for* $\overset{1}{R}{}^i_{jkh}$ *and* $\overset{2}{R}{}^i_{jkh}$ *to be R-recurrent is given by the vanishing of the right hand side of* (2.4)*a and* (2.4)*b respectively.*

Multiplying equations (2.1)a and (2.1)b by g_{ir} we get

$$(2.5)\text{a}\qquad \overset{1}{R}_{jkhr,l}=\lambda_l\overset{1}{R}_{jkhr}+g_{ir,l}\overset{1}{R}{}^i_{jkh}$$

and

$$(2.5)\text{b}\qquad \overset{2}{R}_{jkhr,l}=\lambda_l\overset{2}{R}_{jkhr}+g_{ir,l}\overset{2}{R}{}^i_{jkh},$$

which by definition (2.2) and equations (2.2)e and (2.2)f lead to;

THEOREM 2.3. *If* $\overset{1}{R}{}^i_{jkh}$ *and* $\overset{2}{R}{}^i_{jkh}$ *are R-recurrent tensors then their associates will be R-recurrent iff* $g_{ir,l}\overset{1}{R}{}^i_{jkh}=0$ *and* $g_{ir,l}\overset{2}{R}{}^i_{jkh}=0$, *respectively.*

Differentiating relations $\overset{1}{R}_{jk} \underline{\underline{def}} g^{hr} \overset{1}{R}_{jkhr}$ and $\overset{2}{R}_{jk} \underline{\underline{def}} g^{hr} \overset{2}{R}_{jkhr}$ with respect to x^l covariantly and using (2.2)a, (2.2)b and (2.2)e, (2.2)f we obtain

(2.6)a $$g^{hr}{}_{,l} \overset{1}{R}_{jkhr}=0$$

and

(2.6)b $$g^{hr}{}_{,l} \overset{2}{R}_{jkhr}=0,$$

which leads to;

THEOREM 2.4. *If any two of the following are satisfied:*

i) $\overset{1}{R}_{jk}$ *is R-recurrent* ($\overset{2}{R}_{jk}$ *is R-recurrent*),

ii) $\overset{1}{R}_{jkhr}$ *is R-recurrent* ($\overset{2}{R}_{jkhr}$ *is R-recurrent*),

iii) $g^{hr}{}_{,l} \overset{1}{R}_{jkhr}=0$ ($g^{hr}{}_{,l}\overset{2}{R}_{jkhr}=0$),

then the third is also satisfied.

REMARK. A similar theorem can be established for $\overset{1}{R'}_{jk}$ and $\overset{2}{R'}_{jk}$.

3. Some special cases

If we consider the covariant differentiation due to Berwald of a tensor $T^i_j(x, X)$ and denote it by $T^i_{j(h)}$, Rund [7], then we can easily establish the following:

(3.1) $$\begin{aligned} T^i_{j,h} = {} & T^i_{j(h)} + (\Delta_m T^i_j)(\partial_h X^m - \Delta_h G^m) \\ & + \frac{1}{2} T^m_j \{\overset{1}{S}{}^i_{mh} - X^k \Delta_h \overset{1}{S}{}^i_{mk}\} \\ & - T^i_m \{Y_p \Delta^m \overset{1}{\Gamma}{}^p_{jh} + \frac{1}{2}(\overset{1}{S}{}^m_{jh} - X^k \Delta_h \overset{1}{S}{}^m_{jk})\}. \end{aligned}$$

Since we know that [5]:

(3.2) $$\overset{1}{R}{}^i_{kh} = 2(H^i_{kh} - M^i_{kh}),$$

where

(3.3) $$M^i_{kh} = \{\partial_{[h}(\overset{1}{\Gamma}{}^i_{k]l} X^l) + G^i_{[h|m|}(\overset{1}{\Gamma}{}^m_{k]l} X^l + \overset{1}{\Gamma}{}^m_{k]}) - \overset{1}{\Gamma}{}^m_{[k} \overset{1}{\Gamma}{}^i_{h]m}\},$$

therefore by virtue of (3.1) and (3.2) we obtain on simplification

$$\overset{1}{R}{}^i_{kh,j} = 2[H^i_{kh(j)} + (\Delta_m H^i_{kh})(\partial_j X^m - \Delta_j G^m) + \frac{1}{2} H^m_{kh}(\overset{1}{S}{}^i_{mj} - X^p \Delta_j \overset{1}{S}{}^i_{mp}) - H^i_{km}\{Y_p \Delta^m \overset{1}{\Gamma}{}^p_{hj} + \frac{1}{2}(\overset{1}{S}{}^m_{hj} - X^p \Delta_j \overset{1}{S}{}^m_{hp})\} - H^i_{mh}\{Y_p \Delta^m \overset{1}{\Gamma}{}^p_{kj} + \frac{1}{2}(\overset{1}{S}{}^m_{kj} - X^p \Delta_j \overset{1}{S}{}^m_{kp})\} - M^i_{kh,j}]. \tag{3.4}$$

Now applying equation (2.1)c and the fact that Finsler space F_n is H-recurrent, i.e., it satisfies $H^i_{kh(j)} = \lambda_j H^i_{kh}$, we obtain;

THEOREM 3.1. *If the tensor* $\overset{1}{R}{}^i_{kh}$ *is R-recurrent and* H^i_{kh} *is H-recurrent, then the necessary and sufficient condition for* M^i_{kh} *to be R-recurrent is given by*

$$(\Delta_m H^i_{kh})(\partial_j X^m - \Delta_j G^m) + H^m_{kh} \overset{1}{\Gamma}{}^i_{mj} - H^i_{km} \overset{2}{\Gamma}{}^m_{hj} - H^i_{mh} \overset{2}{\Gamma}{}^m_{kj} - H^m_{kh} G^i_{mj} + H^i_{km} G^m_{hj} + H^i_{mh} G^m_{kj} = 0$$

Since we know that [5]:

$$\overset{1}{R}{}^i_{jkh} = 2(H^i_{jkh} - M^i_{jkh}), \tag{3.5}$$

where

$$M^i_{jkh} = \{\Delta_j \partial_{[h}(\overset{1}{\Gamma}{}^i_{k]l} X^l) - (\Delta_j \overset{1}{\Gamma}{}^i_{m[k}) X^m_{,h]} - \overset{1}{\Gamma}{}^m_{[k} \Delta_j \overset{1}{\Gamma}{}^i_{h]m} - \overset{1}{\Gamma}{}^m_{j[k} \overset{1}{\Gamma}{}^i_{h]m} + G^i_{[h|m|}(2\overset{1}{\Gamma}{}^m_{(k]j)} + X^l \Delta_j \overset{1}{\Gamma}{}^m_{k]l})\}, \tag{3.6}$$

therefore by similar calculation as above we can obtain;

THEOREM 3.2. *If the tensor* $\overset{1}{R}{}^i_{jkh}$ *is R-recurrent and* H^i_{jkh} *is H-recurrent, then the necessary and sufficient condition for* M^i_{jkh} *to be R-recurrent is given by*

$$(\Delta_m H^i_{jkh})(\partial_l X^m - \Delta_l G^m) + H^m_{jkh} \overset{1}{\Gamma}{}^i_{ml} - H^i_{mkh} \overset{2}{\Gamma}{}^m_{jl} - H^i_{jmh} \overset{2}{\Gamma}{}^m_{kl} - H^i_{jkm} \overset{2}{\Gamma}{}^m_{hl} - H^m_{jkh} G^i_{ml} + H^i_{mkh} G^m_{jl} + H^i_{jmh} G^m_{kl} + H^i_{jkm} G^m_{hl} = 0.$$

Since we know that [5]:

$$\overset{2}{R}{}^{l}_{jkh}=\overset{1}{R}{}^{l}_{jkh}+Y_i\Delta^l\overset{1}{R}{}^{i}_{jkh}+L^l_{jkh}, \tag{3.7}$$

where

$$\begin{aligned}L^l_{jkh}=2[&\partial_{[k}Y_i(\Delta^l\overset{1}{\Gamma}{}^{i}_{|j|k]})+\overset{1}{\Gamma}{}^{i}_{j[k}\Delta^l(\partial_{h]}Y_i\\&-(\Delta^l\Delta_m\overset{1}{\Gamma}{}^{i}_{j[k})(Y_i\partial_{h]}X^m-Y_i\overset{2}{\Gamma}{}_{|p|h]}g^{pm})\\&-\Delta_m\overset{1}{\Gamma}{}^{l}_{j[k}(\partial_{h]}X^m-\overset{2}{\Gamma}{}_{|p|h]}g^{pm})-(\Delta_m\overset{1}{\Gamma}{}^{i}_{j[k})\\&Y_i\{\Delta^l\partial_{h]}X^m-\Delta^l(\overset{2}{\Gamma}{}_{|r|h]}g^{rm})\}],\end{aligned}\tag{3.8}$$

therefore differentiating (3.7) covariantly with respect to x^r we obtain

$$\overset{2}{R}{}^{l}_{jkh,r}=\overset{1}{R}{}^{l}_{jkh,r}+Y_{i,r}\Delta^l\overset{1}{R}{}^{i}_{jkh}+Y_i(\Delta^l\overset{1}{R}{}^{i}_{jkh})_{,r}+L^l_{jkh,r}, \tag{3.9}$$

which by virtue of equations (2.1)a and (2.1)b leads to;

THEOREM 3.3. *If* $\overset{1}{R}{}^{i}_{jkh}$ *and* $\overset{2}{R}{}^{i}_{jkh}$ *are R-recurrent, then the necessary and sufficient condition for* L^i_{jkh} *to be R-recurrent is given by*

$$\begin{aligned}&Y_{i,r}\Delta^l\overset{1}{R}{}^{i}_{jkh}+Y_i\{(\Delta^l\lambda_r)\overset{1}{R}{}^{i}_{jkh}-(\Delta^m\overset{1}{R}{}^{i}_{jkh})\Delta^l(\partial_rX^m)\\&-(\Delta_m\overset{1}{R}{}^{i}_{jkh})g^{pl}\overset{2}{\Gamma}{}^{m}_{pr}-(\Delta^l\overset{1}{\Gamma}{}^{i}_{mr})\overset{1}{R}{}^{m}_{jkh}+(\Delta^l\overset{2}{\Gamma}{}^{m}_{jr})\overset{2}{R}{}^{i}_{mkh}\\&+\overset{2}{R}{}^{i}_{jmh}\Delta^l\overset{2}{\Gamma}{}^{m}_{kr}+\overset{2}{R}{}^{i}_{jkm}\Delta^l\overset{2}{\Gamma}{}^{m}_{hr}+g^{pl}_{,r}(\Delta_p\overset{1}{R}{}^{i}_{jkh})\}=0.\end{aligned}$$

Further from equation (3.9) one can easily establish;

THEOREM 3.4. *The necessary and sufficient condition for both* $\overset{1}{R}{}^{i}_{jkh}$ *and* $\overset{2}{R}{}^{i}_{jkh}$ *to be R-recurrent is given by*

$$\begin{aligned}&Y_{i,r}\Delta^l\overset{1}{R}{}^{i}_{jkh}+Y_i\{(\Delta^l\lambda_r)\overset{1}{R}{}^{i}_{jkh}-(\Delta^m\overset{1}{R}{}^{i}_{jkh})\Delta^l(\partial_rX^m)\\&-(\Delta^m\overset{1}{R}{}^{i}_{jkh})g^{pl}\overset{2}{\Gamma}{}^{m}_{pr}-\overset{1}{R}{}^{m}_{jkh}\Delta^l\overset{1}{\Gamma}{}^{i}_{mr}+\overset{2}{R}{}^{i}_{mkh}\Delta^l\overset{2}{\Gamma}{}^{m}_{jr}\\&+\overset{2}{R}{}^{i}_{jmh}\Delta^l\overset{2}{\Gamma}{}^{m}_{kr}+\overset{2}{R}{}^{i}_{jkm}\Delta^l\overset{2}{\Gamma}{}^{m}_{hr}+g^{pl}_{,r}(\Delta_p\overset{1}{R}{}^{i}_{jkh})\}\\&+L^l_{jkh,r}-\lambda_rL^l_{jkh}=0.\end{aligned}$$

DEFINITION 3.1. If in a Finsler space F_n, non-linear connection coefficient $\overset{1}{\Gamma}{}^h_{ik}$ is independent of X^i, then it will be called a *generalised affinely connected space* and will be abbreviated as *GAC- space.*

From the above definition we can observe that F_n is a GAC- space if it satisfies

(3.10) $$\Delta_l \overset{1}{\Gamma}{}^i_{hj}=0,$$

which by virtue of (1.5) implies that

(3.11) $$\Delta_l G^i_{hj}=0.$$

Hence;

THEOREM 3.5. *Every GAC-space F_n is affinely connected but the converse is not true.*

Futher from equation (1.5) for a GAC-space F_n one can easily obtain

(3.12) $$\overset{1}{\Gamma}{}^i_{hj}+\overset{1}{\Gamma}{}^i_{jh}=2G^i_{hj},$$

which together with (1.6) and (3.11) leads to

(3.13) $$\overset{2}{\Gamma}{}^i_{hj}+\overset{2}{\Gamma}{}^i_{jh}=2\Gamma^{*i}_{hj},$$

where Γ^{*i}_{hj} is Cartan's coefficient of connection [7].

In case of a GAC-space F_n, one can easily establish

(3.14) $$g_{ir,l}=0,$$

which together with theorem (2.3) implies;

THEOREM 3.6. *For a GAC-space F_n, if $\overset{1}{R}{}^i_{jkh}$ $(\overset{2}{R}{}^i_{jkh})$ is R-recurrent then $\overset{1}{R}_{jkhr}$ $(\overset{2}{R}_{jkhr})$ is also R-recurrent and conversely.*

REMARKS. i) A theorem similar to above follows from theorem (2.4) also.
ii) In case $\overset{1}{\Gamma}{}^i_{jk}$ is symmetric and the space F_n is affinely connected we can observe that $g_{ir,l}=0$. Thus theorem (3.6) can also be obtained alternatively.

ACKNOWLEDGEMENTS. This work was partially supported by university of Nigeria Research Grant Number 00232/76.

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REFERENCES

[1] Mishra, R.S. and Pande, H.D.: *Recurrent Finsler spaces*, J. Ind. Math. Soc., 1968, 32, 17—32.

[2] Misra, R.B. and Mishra, R.S.: *Curvature tensor arising from non-linear connections in a Finsler space*, Bull. Acad. Pol. des Sci., 1969, 17, no. 11, 755—760.

[3] Misra, R.B.: *On a recurrent Finsler space*, Rev. Roum. Math. Pures et Appl. 1973, 18, No. 5, 701—712.

[4] Móor, A.: *Untersuchungen über Finsler-Raume Von rekurrenter Krummung.*, Tensor, N.S., 1963, 13, 1—18.

[5] Rastogi, S.C.: *On some curvature properties in a Finsler space based on non-linear connection* (Under publication).

[6] Rund, H.: *Some remarks concerning the theory of non-linear connections*, Nederl. Akad. Wetensch. Proc., Ser. A., 1958, 61, 341—347.

[7] Rund, H.: *The differential geometry of Finsler spaces*, Springer-Verlag, 1959.

[8] Ruse, H.S.: *Three dimensional spaces of recurrent curvature*, Proc. London. Math. Soc., 1949, 50, no. 2, 438—446.

[9] Sen, R.N.: *Finsler spaces of recurrent curvature*, Tensor, N.S., 1968, 19, 291—299.

[10] Walker, A.G.: *On Ruse's spaces of recurrent curvature*, Proc. London. Math. Soc., 1951, 52, no. 2. 36—64.