Kyungpook Math. J.
Volume 20, Number 1
June, 1980

SUBSETS WITH NO CLUSTER POINTS

By Y. L. Lee

The purpose of this paper is to prove that in a manifold of dimension ≥ 2, for any two sets A and B having no cluster points and having same condinality, then there exists homeomorphism f such that $f(A)=B$. Then we use this property to study the topology $\mathscr{V}$ with the same class of homeomorphisms $H(Z,\mathscr{U})=H(X,\mathscr{V})$. By manifold we always mean separable connected manifold without boundary.

THEOREM 1. *Let X be a manifold with dimension ≥ 2 and A, B are subsets of X with no cluster point and have same condinality. Then there exists a homeomorphism f of X onto itself such that $f(A)=B$.*

PROOF. Since A and B have no cluster points, the condinality of A and B is at most countable. If A and B have same finite number of points. Then by the homogeneity of X, we have a homeomorphism f of X onto itself such that $f(A)=B$. If A and B are countable, let $A=\{a_1, a_2, \ldots, a_n, \ldots\}$ and $B=\{b_1, b_2, \ldots, b_n, \ldots\}$

First choose an open connected set U_1 such that $\{a_1, b_1\}\subseteq U_1$ and $\mathrm{cl}(U_1)\cap(\{a_2,\ldots,a_n,\ldots\}\cup\{b_1,\ldots,b_n,\ldots\})=\phi$ and $X\backslash U_1$ is a connected manifold and a homeomorphism f_1 with $f_1(a_1)=b_1$ and $f_1(x)=x$ for $x\notin U_1$. After constructing U_{n-1}, choose U_n to be an open connected set such that

$$\mathrm{cl}(U_n)\cap(\mathrm{cl}(U_1)\cup\ldots\cup\mathrm{cl}(U_{n-1})\cup\{a_{n+1}, a_{n+2}\ldots\}\cup\{b_2,\ldots,b_n,\ldots\})=\phi$$

and $X\backslash\cup\{\mathrm{cl}(U_i)\,|\,i=1,2,\ldots,n\}$ is a connected manifold. In this way we constructed a sequence of open connected sets and sequence of homeomorphism $\{f_n\}$ such that f_n is fixed outside U_n and $f_n(a_n)=b_n$. Then let f be the function defined on X such that $f(x)=f_n(x)$ if $x\in U_n$ and $f(x)=x$ if $x\notin U_n$ for all n. Then f is a homeomorphism and $f(A)=B$.

This result is useful in studying the classes of homeomorphisms.

THEOREM 2. *Let $(Z,\mathscr{U})$ be a manifold and $H(Z,\mathscr{U})$ the class of homeomorphisms of $(X,\mathscr{U})$ onto itself. Let $\mathscr{V}=\{U\in\mathscr{U}\,|\,U=\phi$ or $X\backslash U$ has no cluster point.$\}$*

then $\mathscr{V}$ is a topology in X and $H(X,\mathscr{U})\subsetneq H(X,\mathscr{V})$

PROOF. It is clear that $\mathscr{V}$ is a topology and $H(X,\mathscr{U})\subset H(X,\mathscr{V})$. To see thet $H(X,\mathscr{U})\neq H(X,\mathscr{V})$, take a point $p\in X$ and an open ball U with center p. Let f be a function which makes a rotation on U on any direction of angle between 0 and π, and fixed outside U, then $f\in H(X,\mathscr{V})$ but $f\notin H(X,\mathscr{U})$.

COROLLARY. *Let $(X,\mathscr{U})$ be a manifold and $\mathscr{V}\subset\mathscr{U}$ is a topology on X. If $H(X,\mathscr{U})=H(X,\mathscr{V})$, then there exists $V\neq\phi$ in $\mathscr{V}$ such that $X\backslash V$ contains cluster points.*

PROOF. Let $\dim(X,\mathscr{U})\geq 2$. Then there exists $\phi\neq V\in\mathscr{V}$ with $X\backslash V$ contains infinitely many points. Because otherwise, by theorem 1 $\mathscr{V}=\{V\in\mathscr{U}\mid V=\phi$ or Card $(X\backslash V)\leq m\}$ for some positive integer m. Then $H(X,\mathscr{V})$ would be the set of all one to one functions of X onto itself. If $X\backslash V$ contains no cluster points for every non-void V in $\mathscr{V}$ then by theorem 1 again $\mathscr{V}=\{V\in\mathscr{U}\mid V=\phi$ or Card $(X\backslash V)\leq\aleph_0$ and $X\backslash V$ has no cluster points$\}$. By theorem 2, $H(X,\mathscr{V})\supsetneq H(X,\mathscr{U})$.

If $\dim(X,\mathscr{U})=1$, and X is a circle, and for every non-void $V\in\mathscr{V}$ $X\backslash V$ contains no cluster point, then $X\backslash V$ is a finte set, it is easy to see $\mathscr{V}=\{V\in\mathscr{U}\mid V=\phi$ or $\text{Card}(X\backslash V)\leq m\}$ for some positive integer m and $H(X,\mathscr{U})\subsetneq H(X,\mathscr{V})=$the set of all one-one onto maps. Hence there is $V\neq\phi$ with $X\backslash V$ contains infinitely many points. Since X is compact, $X\backslash V$ has cluster points. If X is a real line, by similar argument, there exists $V\neq\phi$ in $\mathscr{V}$ such that Card $(X\backslash V)=\aleph_0$. If $X\backslash V$ has no cluster point and $X\backslash V$ is unbounded in both side, then it is easy to see that there exist $f\in H(X,\mathscr{U})$ with $f(X\backslash V)=$the set of all integers. Also there exist $g\in H(Z,\mathscr{U})$ such that

$$g(X\backslash V)=\{1,2,3,\dots\}\cup\left\{-\frac{1}{2},\ -\frac{3}{2},\ -\frac{5}{2},\dots\right\}$$

Hence $f(X\backslash V)\cap g(Z\backslash V)=$the set of all positive integers which is closed $(Z,\mathscr{V})$. If $Z\backslash V$ is unbounded in one side, then there exist f,g in $H(Z,\mathscr{U})$ such that $f(X\backslash V)=$the set of all non-negative integers $g(Z\backslash V)=$the set of all non-positive integers. Hence the set of all integers is closed in $(X,\mathscr{V})$ and $\mathscr{V}\supset\{V\in\mathscr{U}\mid V=\phi$ or $X\backslash V$ has no cluster points.$\}$

However, if $\mathscr{V}$ does not contain any non-void V with $X\backslash V$ having cluster points then $H(X,\mathscr{V})\subsetneq H(X,\mathscr{U})$. This contraction proves that there exists $V\neq\phi$ in $\mathscr{V}$ with $X\backslash V$ contains cluster points.

Kansas State University