HONAM MATHEMATICAL JOURNAL
Volume 2, Number 1, July 1980

A NOTE ON G-VECTOR BUNDLES

YANG-KON KIM

ABSTRACT

우리는 먼저 Principal G-bundle의 성질을 살피고 representation of G over C를 irreducible CG-space의 direct sum으로 표시하여 Schur's Lemma를 이용하면 E가 임의의 CG-space, $\sigma E = Hom_c(E\sigma, E)$라 할 때

$$\oplus_\sigma (E_\sigma \otimes \sigma E) \longrightarrow E$$

가 G-ismorphism이 됨을 알아본다.

본 논문의 목적은 이러한 결과를 이용하여 $K(X)$와 $K_G(X)$의 관계를 구명하는 데 있다. (Theorem8—10)

§1. Introduction

The concept of a G-vector bundle is very important in K-theory and representation theory. In this note we introduce the concept of a G-vector bundle and define the ring $K_G(X)$, where G is a compact Hausdorff topological group and X is a G-space over the complex field C. we shall also study some relations betwen $K(X)$ and $K_G(X)$. In details, we shall define some concepts in §2, and deal with the properties of Haar measure, invariant positive definite Hermitian form under G and representation of G over C in §3. Finally, we shall deal with some relations between $K(X)$ and $K_G(X)$ (Theorem 8—10)

§2. Definitions

Let Top be the category consisting of topogical spaces and continuous maps, and let F be one of the classical fields R(the real numbers), C(the complex numbers).

If we denote the set of isomorphic classes of F-vector bundles over X by $Vect_F(X)$, then with direct sum and tensor product $Vect_F(X)$ becomes a semiring. Therefore if Sr is the category of semirings and semiring homomorphisms, then $Vect_F: Top$

$\rightarrow Sr$ is a contravariant functor. That is, for any $[\xi]$, $[\eta] \in Vect_F(X)$, $[\xi \oplus \eta]$ and $[\xi \otimes \eta]$ are in $Vect_F(X)$, and for any $f: Y \rightarrow X$ $Vect_F(f)([\xi]) = [f^*(\xi)]$, where $f^*(\xi)$ is the induced bundle by f. The ring completion of a semiring S is a pair (S^*, θ), where S^* is a ring and $\theta: S \rightarrow S^*$ is a morphism of semirings such that if $f: S \rightarrow R$ is any morphism into a ring there exists a unigue ring morphism $g: S^* \rightarrow R$ such that $g\theta = f$, where R is an arbitrary ring.

$$\begin{array}{ccc} S & \xrightarrow{\theta} & S^* \\ & {}_{f}\searrow \quad \copyright \quad \swarrow_{\exists ! g} & \\ & R & \end{array}$$

For the construction of S^* we consider pairs $(a, b) \in S \times S$ and put the following equivalence relation on these pairs; that is, (a, b) and (a', b') are equivalent provided there exists $c \in S$ such that $a + b' + c = a' + b + c$. Let $\langle a, b \rangle$ denote the equivalence class of (a, b), and S^* the set of equivalence classes $\langle a, b \rangle$. If we define $\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle$ and $\langle a, b \rangle \cdot \langle c, d \rangle = \langle ac + bd, bc + ad \rangle$, then S^* is a ring. Finally $\theta: S \rightarrow S^*$ is defined by $\theta(a) = \langle a, o \rangle$. The $K_F(X)$ ring of a space X is the ring completion of $Vect_F(X)$. Therefore if g is the category of groups and homomorphisms, then K_F: $Top \rightarrow$ g is a contravariant functor. In consequence we have the following commutative diagram

$$\begin{array}{ccc} Vect_F(X) & \xrightarrow{Vect_F(f)} & Vect_F(Y) \\ \downarrow \theta & \copyright & \downarrow \theta \\ K_F(X) & \xrightarrow[K_F(f)]{} & K_F(Y) \end{array}$$

for any map $f: Y \longrightarrow X$ in Top.

Let $X \in Top$ and G be a toplogical *group*. X is called a G space if there exists a continuous map $G \times X \rightarrow X$ satisfying the associative condition for all $g_1, g_2 \in G$ and $x \in X$. A G-map between two G-spaces is a continuous map $f: Y \rightarrow X$ such that for all $g \in G$ and $y \in Y$, $f(gy) = gf(y)$. More generally, if X is G-space and Y is an H-space and $\theta: H \rightarrow G$ is a continuous homomorphism, we say that $f: Y \rightarrow X$ is a θ-equivariant map if it is continuous and if $f(hy) = \theta(h) f(y)$ for all $h \in H$, $y \in Y$.

A G-vector bundle over the G-space X is a vector bundle $P: E \rightarrow X$ such that (1) E is a G-space (2) p is a G-map (3) *if* $g \in G$, $g: p^{-1}(x) \rightarrow p^{-1}(gx)$ is a linear map. (Note that $g: X \rightarrow X$ is defined by $g(x) = gx$ for all $x \in X$ and $g: p^{-1}(x) \rightarrow p^{-1}(gx)$ is defined by $g(v) = gv$ for all $v \in p^{-1}(x)$.

A G-homomorphism $\xi \to \eta$ between two G-vector bundles ξ and η is a map which is boch a vector bundle homomorphism and a G-map. If for every pair $(x,y) \in X \times X$ there exists an element $g \in G$ such that $y = gx$, then x and y are said to be G-equivariant. In this case, there is the canonical projection $X \to X/G$, where we have to note that X/G is also a G-space. Let X bea G-space X is said to be effective if for $g \in G$ and $x \in X$ $gx = x$ implies that $g = e$, where e is the identity of G. Let $X^* = \{(gx, x) \in X \times X\}$. $\tau : X^* \to G$ is a translation function if for all $(gx, x) \in X^*$ $\tau(gx, x) = g$. This translation function τ has the following properties (1) $\tau(x,x) = 1$ (2) $\tau(g_2 g_1 x,\ g_1 x)\ \tau(g_1 x, x) = \tau(g_2 g_1 x, x)$ (3) $\tau(gx, x) = (\tau(x, gx))^{-1}$.

A G-space X is called principal if X is effective with a continuous translation function $\tau : X^* \to G$. A vector bundle (X, p, B) is a principal G-bundle if(X, p, B) is a G-vector bundle and X is a principal G-space. If $q : X \to X/G$ is the canonical projection, then by the universal property of the induced bundle there is a unique G-vector bundle structure on q^*F. Hence we have that q^* is a functor from the category of vector bundles and homomorphisms over X/G to the category of G-vector bundles and G-homomorphisms over X.

proposition 1. If X is a principal G-bundle then q^* is an equivlaence

proof: we have to find a functor r such that both q^*r and rq^* are naturally isomorphic to the identity functor. Given a G-vector Bundle E over X we define $r(E)$ to be the map $E/G \to X/G$.

we first prove that this is a vector bundle over X/G; the only problem is to show that $E/G \to X/G$ has local triviality. Let F be the field of scalars and n the fibre dimension of E. Let U be a neighborhood of a given point in X/G such that $X \to X/G$ is trivial over U. We need only to show that the restriction of $E/G \to X/G$ to U is locally trivial. Let $x' \in X'$ where$X = G \times X'$. There is a neighborhood V of the identity $e \in G$, and a neighborhood W of x' such that $p : E \to G \times X'$ is locally trivial over$V \times W$. Then there is an isomorphism of vector bundles over $e \times W$

$e \times W \times F^n \to p^{-1}(e \times W)$ which can be extended uniguely to a G-isomorphism $G \times W \times F^n \to p^{-1}(G \times W)$ of G-vector bundles over $G \times W$. This induces an isomorphism of vector bundles between $W \times F^n \to W$ and restriction to W of $E/G \to X'$. Thus $E/G \to X/G$ is a vector bundle $r(E)$ over X/G.

The commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E/G \\ \downarrow & © & \downarrow \\ X & \longrightarrow & X/G \end{array}$$

shows that $E \to X$ is the induced bundle, and hence q^*r is isomorphic to the identity. To show that rq^* is isomorphic to the identity we write $q^*E' = X \times_{X/G} E' \subset X \times E'$. G acts on the lst factor. The map $q^*E' \to E'$ induces an isomorphism $q^*E'/G \to E'$ of vector bundles over X/G. Q.E.D.

COROLLARY 2. The category of G-vector bundles over G is eguivalent to the category of vector spaces.

Let H be a subgroup of G, and Y an H-space. Let $G \times_H Y$ denote the identification space obtained from $G \times Y$ by the equivalence relation: $(g_1, y_1) \sim (g_2, y_2)$ iff $\exists h \in H$ such that $g_2 = g_1 h^{-1}$ and $y_2 = h y_1$. Then $G \times_H Y$ admits a G-space structure; we define
$g(g_1, y) = (gg_1, y)$

PROPOSJTJON 3. The category of H-vector bundles over Y is equivalent to the category of of G-vector bundles over $G \times_H Y$.

PROOF: consider the subspace $H \times_H Y$ of $G \times_H Y$. Clearly this space Is homeomorphic to Y. Therefore any G-vector bundle over $G \times_H Y$ defines by restriction an H-vector bundle over $H \times_H Y = Y$. Now let E' be an H-vector bundle over Y and consider the map $G \times_H E' \to G \times_H Y$. Here $G \times_H E$ is a G-space, and also a vector bundle. The restriction of $G \times_H E'$ to $H \times_H Y$ is clearly $H \times_H E' = E'$.

On the other hand if E is a G-vector bundle over $G \times_H Y$, then there is a map $G \times_H (E/H \times_H Y) \to E$ defined by $(g, e) \mapsto ge$, which is a G-ismorohism. Q.E.D.

§ 3. Invariant Integrals

A FG-space is a finite dimension vector space V over F provided with a continuous homomorphism $\theta: G \to Aut\ V$. Such a V is also called a representation of G over F or a G-space over F. Given two G-spaces over F, V and W, we can form $Hom(V, W)$ the set of F-linear maps from V to W.

We shall restrict ourselves to complex representations (i,e, $F = C$). If we put $V^* = Hom_c(V, C)$ for a G-space V over C then we have $Hom_c(V, W) \cong V^* \otimes_c W$.

Let G be a compact topological group.

Then for each continuous function $f: G \to R$(the set of all real numbers) we can define a real number $\int_G f = \int_{g \in G} f(g) dg$ so as to satisfy the following conditions. (1) $\int_G$ has the usual properties of an integral, i.e., it is a positive linear functional. (2) $\int_G 1 = 1$ (3) The integral is invariant under right and left translations; i.e., for each $g \in G$ we have

$$\int_{h\in G} f(gh)dh=\int_{h\in G} f(h)dh=\int_{h\in G} f(hg)dh$$

The above integral is called an invariant integral.

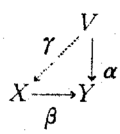
PROPOSITION 4. Let G be compact and Hausdorff topological group and let V be a G-space over C. Given a representation $\theta:G\to Hom_c(V,V)$, then $I=\int_G \theta\in Hom_c(V,V)$ is idempotent ($I^2=I$) and its image is the subspace of elements invariant under G.

PROOF: For a fixed element $v\in V$, $Iv=v\int_{g\in G}\theta(g)dg=\int_{g\in G}v\theta(g)dg=\int_{g\in G}gvdg$ were we put $\theta(g)=g$. Then for $g'\in G$ $g'(IV)=g'\int_{g\in G}gvdg=\int_{g\in G}g'(gv)dg=\int_{g\in G}gvdg=Iv$ by the invariant integration. Furthermore $Iv=\int_{g\in G}gvdg=v\int_{g\in G}dg=v$. Q.E.D.

Let V be a G-space over C. A structure map on V is a G-map $j:V\to V$ such that (1) j is conjugate linear, that is, $j(zv)=\bar{z}j(v)(z\in c)$, and (2) $j^2=\pm 1$. Let $\phi:V\times V\to C$ be a Hermitian form. Then $\psi(x,y)=\int_{g\in G}\phi(g^{-1}x,\ g^{-1}y)dg$ is a positive definite Hermitian form invariant under G. Moreover, if V carries a structure map j, we can choose ψ so that $\psi(jx,jw)=\overline{\psi(x,y)}$. We define tV *to* have the same underlying set as V and the same operations from G, but we make C act in a new way: z acts on tV as $\bar{z}$ used to act on V. Using an invatiant positive definite Hermitian form under G, $V^*\cong tV$.

PROPOSITION 5. If G is a compact group, then every CG-space V is projective.

PROOF: We have to find a CG-map y: $V\to X$ such that the following diagram commutes



where β is onto and α is any CG-map. $Hom_c(V,X)$ is a CG-space by $(gh)(v)=g(h(g^{-1}v))$ for $g\in G$, $h\in Hom\ (V,X)$ and $v\in V$. Then, by proposition 4, if we take any C-map $\delta:V\to X$ and integrate $g\delta$, we get a C-map $\gamma:V\to X$ which is invariant, that is, a CG-map. It is given by $\gamma=\int_{g\in G}(\theta_X g)\delta(\theta_V g^{-1})$

where $\theta_X:G\to Hom_c(X,X)$ and $\theta_V:G\to Hom_c(V,V)$. Since we can choose δ to be a C-map such that $\beta\delta=\alpha$, then we have

$$\beta\gamma=\beta\int_{g\in G}(\theta_X g)\delta(\theta_V g^{-1})=\int_{g\in G}\beta(\theta_x g)\delta(\theta_V g^{-1})=\int_{g\in G}(\theta_Y g)\beta\delta(\theta_v g^{-1})$$

$$=\int_{g\in G}(\theta_Y g)\alpha(\theta_V g^{-1})=\int_{g\in G}\alpha=\alpha.\qquad \text{Q.E.D.}$$

A nonzero CG-space V is reducible if some proper subspace of V is a CG-space, otherwise V is irreducible.

THEOREM 6. If G is a compact group, every CG-space V is the direct sum of irreducible CG-spaces.

PROOF: By induction over $dim_c(V)$; for CG-spaces W with $dim_c(W) < dim_C(V)$ we assume that our statement is true. It will be sufficient to show that if V is reducible, then it is the direct sum of two subspaces of less dimension. Suppose that V has a proper subspace S which is a CG-space, then proposition 5 shows that the exact seguence

$$O \longrightarrow S \longrightarrow V \longrightarrow V/S \longrightarrow O$$

splits, so we have a CG-isomorphism $V \cong S \oplus V/S$. In particular imposing on V an invariant Hermitian from under G we can take V/S as the orthogonal complement T of S, and thus $V \cong S \oplus T$. Without proof we describe Schur's lemma.

LEMMA 7. Let G be any topological group.

(1) If $f: V \to W$ is FG-map and V, W are irreducible then f is either zero or an isomorphism.

(2) If $f: Y \to Y$ is a CG-map and V is irreducible then $f(v) = \lambda v$ for some constant $\lambda \in C$.

If V is a CG-space and $gv = v$ for all $g \in G$ and $v \in V$, then we say that V is trivial CG-space. Now let G be a compact Hausdorff group and E be a G-vector space over C. In other words E is a representation of G over C. Using the above invariant integral on G and G-invariant Hermitian form on E, we can write $E = \oplus i \; n_i \; E_i$

where E_i is an irreducible CG-space and $n_i E_i$ denotes the direct sum of n_i copies of E_i. The integers n_i are determined uniguely by E. By Schur's lemma we have, for an irreducible CG-space E, that the $End_G \; E = Hom_c(E, E) = C$.

Let $\{E_\sigma\}_{\sigma \in S}$ be a complete set of irreducible inequivalent representations of G over C. That is, if $\sigma \neq \tau$ in S then $E_\sigma \not\equiv E_\tau$. In this case S is a countable set. If E is any CG-space we write $\sigma E = Hom_c(E_\sigma, E)$. There is a G-map $\oplus_\sigma(E\sigma \otimes \sigma E) \to E$ which by the above results is a G-isomorphism.

§4. MAIN THEOREMS

We now come to the definition of the ring $K_G(X)$. Let X be a compact G-space. The equivalence classes of G-vector bundles over X form a commutative semigroup $Vect_G(X)$. There is a semigronp homomorphism θ: $Vect_G(X) \to K_G(X)$ such that, given any semigroup homomorphism φ: $Vect_G(X)\to H$ form $Vect_G(X)$ to a graph H, there is a unique group homomorphism ψ: $K_G(X)\to H$ such that$\psi\theta=\varphi$. Hence $K_G(X)$ is the quotient of the free abelian group on $Vect_G(X)$ by the subgroup generated by elements of the form $E\oplus F$-E-F. The tensor product of two G-vector bundles becomes a G-vector bundle if we allow G to act by the diagonal action. This introduces a commutative bilinear product into the free abelian group on $Vect_G X$, and induces a commutative bilinear product in $K_G(X)$. Thus $K_G(X)$ is a commutative ring with unit represented by the G-vector bundle $X\times C\to X$ where G acts trivially on the field C. If $G=\{e\}$ we obtain the ring $K_C(X)$ in §2. If X is a point we obtain the ring $R(G)$ of virtual representations. We recall that if G is compact then $R(G)$ is the free abelian group on the irreducible representations.

The G-map from X to a point induces a ring homomorphism $R(G)\to K_G(X)$, which maps a vector space V to the G-vector bundle $X\times V\to X$. Here the map is the product projection and the action of g is given by $g(x,v)=(gx,gv)$.

Note that this homomorphism makes $K_G(X)$ an $R(G)$-module. we have a category whose objects are pairs (G,X) where X is a G-space, and whose morphisms are pairs $(\theta,f):(H,Y)\to(G,X)$ where θ:H$\to G$ is a homomorphism and $f:Y\to X$ is θ-equivariant. Given a G-vector bundle on X, (θ,f) induces an H-vector bundle over Y. Hence K_G defines a contravariant functor from the above category to the category of commutative rings with unit.

Thus the map $(G,X)\to(G,pt)$ induces the ring homomorphism $R(G)\to K_G(X)$ mentioned above where pt is a point. The obvious map $(G,X)\to(e,X/G)$ induces a ring homomorphism $K(X/G)\to K_G(X)$ which, when X is trivial G-space, becomes a ring homomorphism $K(X)\to K_G(X)$

THEOREM 8. If X is a principal G-bundle, $K(X/G)\to K_G(X)$ is an isomorphism.

PROOF. By proposition 1 and sincethe map $(G,X)\to(e,X/G)$ induces a ring homomorphism $K(X/G)\to K_G(X)$.

THEOREM 9. If X is a trivial CG-space and G is compact and Hausdorff, then the composite $R(G)\otimes K(X)\to K_G(X)\otimes K_G(X)\to K_G(X)$ is an isomorphism.

PROOF. we have a map $Vect_G(X)\to R(G)\otimes K(X)$ which, in the notation of the end of §3, sends a G-vector bundle E over X to $\Sigma_\rho(E_\rho\otimes_\rho E)$. This factors to give an additive homomorphism $K_G(X)\to R(G)\otimes K(X)$.

The composite $Vect_G(X)\to K_G(X)\to R(G)\otimes K(X)\to K_G(X)$ is the canonical projection $Vect_G(X)\to K_G(X)$ as we have seen from §3. Therefore $K_G(X)\to R(G)\otimes K(X)\to K_G(X)$

is the identity. The composite $R(G)\otimes K(X)\to K_G(X)\to R(G)\otimes K(X)$ is easily to be seen the identity by checking on elements of the form $E_e\otimes F$ where F is a vector bundle over X. Q.E.D.

THEOREM 10. Let H be a subgroup of G.

The map $(H,Y)\to(G, G\times_H Y)$ defines an isomorphism $K_G(G\times_H Y)\to K_H(Y)$ for every H-space Y.

PROOF. By proposition 3.

REFERENCES

1. J.F. Adams: *Lecture on Lie Groups*, W.B. Benjamin I.N.C. (1969).
2. M. Atyah: *K-Theory*, idid (1964).
3. M. Atyah: *Equivalent K-Theory*, Univ. of Warwick coventry (1970).
4. R. Bott: *Lectures on K(X)*, W.B. Benjamin I.N.C. (1969).
5. L. Pontrajagin: *Topological Groups*, Princeton Univ. (1949).

(*Jeon-Ju Woo Suk Women's College*)