HONAM MATHEMATICAL JOURNAL
Volume 2, Number 1, July 1980

ON THE PROPERTIES OF LOCAL HOMOLOGY GROUPS OF SHEAVES*

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ABSTRACT

모든 記號는 G.E Bredon의 著 Sheaf Theory의 記號를 따른다. A가 torsion free이며 elementary sheaf이라 하자. 그리고 L을 injective L-module이라 하자 $dim_\varphi X<\infty$이라면 support의 familyφ와 locally subset z에 대하여

$$\Gamma_Z(\tilde{\ }Hom(\Gamma_\varphi(L),L)\otimes A)\simeq H_0{}^Z(X:A)$$

$$H_{-p}^Z(X:A)=0,\quad p=1,2,3,\cdots\cdots$$

이며 support의 family c 와 compact subset z에 대하여도

$$\Gamma_Z(\tilde{\ }Hom(\Gamma_c(L),L)\otimes A)\simeq H_0{}^Z,(X:A)$$

$$H_{-p}^Z(X:A)=0,\quad p=1,2,3.\cdots\cdots$$

A가 elementary이면 locally closed z와 z에서 closed인 z' 그리고 $z''=z-z'$에 대하여 exact sequence

$$\cdots\cdots\to H_p^{z'}(X:A)\to H_p^z(X:A)\to H_p^{z''}(X:A)\to\cdots\cdots$$

가 存在한다.

Throughout this paper, we assume that X is topological space and all sheaves are sheaves of L-modules on X without any statements where L is a principal ideal domain. For each open set U of X, $\Gamma_\varphi(A)$ is the set of all sections of sheaf A with supports in φ where φ is a family of supports. We shall use the notations of sheaf theory [1].

A sheaf A on X is said to be elementary if it is locally constant with finitely generated stalks. A will be called φ-element if in addition, each member of φ can be covered by a finite number of open subsets of X such that A is constant on each menber of this open covering.

* This research was supported by the Korean Ministry of Education Scholarship Foundation for 1979.

LEMMA 1. If a sheaf A is flabby and torsion free and sheaf μ is elementary, then $A\otimes\mu$ is flabby.

PROOF. Since flabbiness is local properties, we can assume that μ is constant without loss of generality. If μ is a sheaf of finitely generated free L-module, we can put

$$\mu=X\times L^n$$

where $L^n=L\times L\times\cdots\cdots\times L$ has the discrete topology. This means that μ is flabby. For open U of X, $A(X)\to A(U)$ and $\mu(X)\to\mu(A)$ are surjective and since A is torsion free

$$A(X)\otimes\mu(X)\longrightarrow A(U)\otimes\mu)X)$$

and $$A(U)\otimes\mu(X)\longrightarrow A(U)\otimes\mu(U)$$

are surjective, Therefore

$$(A\otimes\mu)(X)\longrightarrow)(A\otimes\mu)(U)$$

is surjective.

Let μ be a constant sheaf of finitely generated L-modules. Then there is a positive integer n such that $X\times L^n\to\mu(X)$ is surjective.

From the commutative diagram

$$\begin{array}{ccccc} X\times L^n & \longrightarrow & \mu(X) & \longrightarrow & 0 \text{ (exact)} \\ \downarrow & & \downarrow & & \\ U\times L^n & \longrightarrow & \mu(U) & \longrightarrow & 0 \text{ (exact)} \end{array}$$

$\mu(X)\longrightarrow\mu(U)$ is surjective for all open U in X. Thus $A\otimes\mu$ is flabby.

Let $\mathfrak{A}$ be a cosheaf on X and L be L-module. Then there is the induced L-homomorphism

$$\rho_{uv}\colon Hom(\mathfrak{A}(U),\ L)\longrightarrow Hom(\mathfrak{A}(V),\ L)$$

from $i_{u,v}\colon \mathfrak{A}(V)\longrightarrow\mathfrak{A}(U)$ for open $V\subset U$.

Thus $U\to Hom(\mathfrak{A}(U).\ L)$ and ρ_{uv} make a presheaf.

Let $\tilde{}Hom(\mathfrak{A}.L)$ be the sheaf generated by this presheaf. Then $\Gamma\tilde{}Hom\ (\mathfrak{A},L)=Hon(\mathfrak{A}(X),L)$.

Let $\mathfrak{A}_*$ be a graded cosheaf with a differential d: $\mathfrak{A}_n\longrightarrow\mathfrak{A}_{n-1}$ of degree -1 such that $d_2=0$.

Consider the differential sheaf

$$D(\mathfrak{A}_*, L) = {}^{\sim}Hom(\mathfrak{A}_*,\ L^*)$$

where L^* is the canonical injective resolution of L and

$$D^n(\mathfrak{A}_*, L) = \sum_{p+q=n} {}^{\sim}Hom(\mathfrak{A}_p,\ L^q).$$

The differential $d: D^n \to D^{n+1}$ is $d' - d''$ where d′ is the homomorphism induced by the differential $L^q \longleftarrow L^{q+1}$ and $(-1)^n$ d'' is the homomorphism induced by $\mathfrak{A}_{p+1} \longrightarrow \mathfrak{A}_p$. If α^* is a c-soft differential cosheaf with the gradation

$$(\Gamma_c \alpha^*)_n = \Gamma_c \alpha^{-n}$$

The differential sheaf $D(\Gamma_c \alpha^* : L)$ will also be denoted by $D(\alpha^*)$. Also, as above, we let D_n stand for D^{-n}.

Consider the canonical injective resolution $I^*(X:L)$ of L

For a sheaf A on X, we define

$$C^\varphi_*(X:A) = \Gamma_\varphi[D_*(I^*(X:L)) \otimes A]$$

$$H^\varphi_p(X:A) = H_p(C^\varphi_*(X:A))$$

with respect to family φ of supports.

LEMMA 2. Let $0 \longrightarrow \mu' \longrightarrow \mu \longrightarrow \mu'' \longrightarrow 0$

be an exact sequence of sheaves. If μ' is elemeutary then there is the long exact sequence of homology groups

$$\cdots\cdots \to H^\varphi_p(X:\mu') \longrightarrow H^\varphi_p(X:U) \longrightarrow H^\varphi_p(X:\mu'') \to \cdots\cdots$$

for any family φ of supports.

PROOF. We have already know that $D(I^*(X:U))$ is flabby and torsion free. Hence the sequence

$$0 \text{---} D(I^*(X:L)) \otimes \mu' \longrightarrow D(I^*(X:L)) \otimes \mu \longrightarrow D(I^*(X:L)) \otimes \mu'' \longrightarrow 0$$

of sheaves is exact. By Lemma 1, $D(I^*(X:L)) \otimes \mu'$ is flabby.

Therefore

$$0 \longrightarrow \Gamma_\varphi(D(I^*(X:L)) \otimes \mu') \longrightarrow \Gamma_\varphi(D(I^*(X:L)) \otimes \mu) \longrightarrow \Gamma_\varphi(D(I^*(X:L)) \otimes \mu'') \longrightarrow 0$$

is exact for any family φ of supports. By the property of homology functor we have the long exact sequence

$$\cdots\cdots\rightarrow H_p^{\varphi}(X:\mu')\longrightarrow H_p^{\varphi}(X:\mu)\longrightarrow H_p^{\varphi}(X:\mu'')\longrightarrow\cdots\cdots$$

Let z be a locally closed subspace of X. We define

$$\Gamma_Z(A)=\{s\in A(X)\mid |s|\subset z\}\quad \text{for a sheaf } A \text{ on } X \text{ and}$$
$$\Gamma_Z(A|U)=\{s\in A(U)\mid |s|\subset z\}\quad \text{for open } U \text{ with } z\subset U$$

For a locally closed subset z of X and a sheat A on X, we define

$H_p^Z(X:A)=H_p(\Gamma_Z(C_*(X:A)))$ is local cohomology groups of X with

coefficients in A and supports in z. By Lemma 2, the following statement is obvious.

Let $0\longrightarrow\mu'\longrightarrow\mu\longrightarrow\mu''\longrightarrow 0$ be exact sequence of sheaves and z a locally closed subset of X. If μ' is elementary then there is a long exact sequence.

$$\cdots\cdots\rightarrow H_p^Z(X:\mu')\longrightarrow H_p^Z X:\mu)\longrightarrow H_p^Z(X:\mu'')\longrightarrow\cdots\cdots$$

For any family φ of supports, we define $dim_\varphi X$ to be the least integer n (or ∞) such that $H_\varphi^k(X:A)=0$ for all sheaves and for all $k>n$.

THEOREM 3. Let A be a torsion free sheaf which is elementay and let L be injective as L-module.

i) If $dim_\varphi X<\infty$, then for any family φ of supports and locally closed subset z of X

$$\Gamma_Z(\tilde{\ }Hom(\Gamma_\varphi(L),L)\otimes A)\simeq H_0^Z(X:A)$$
$$H_{-p}^Z(X:A)=0,\qquad p=1,2,\cdots\cdots$$

ii) For the family c of supports and a compact subset z of X

$$\Gamma_Z(\tilde{\ }Hom(\Gamma_c(L),L)\otimes A)\simeq H_0^Z(X:A)$$
$$H_{-p}^Z(X:A)=0,\qquad p=1,2,\cdots\cdots$$

PROOF. ii) Since A is elementary iff A is C-elementary

$$H_{-p}^c(X:A)=H_{-p}(\Gamma_c\ D(I^*)\otimes A$$

for an injective resolution $0\longrightarrow L\longrightarrow I^*$ of L. Since L is injective $0\longrightarrow L\longrightarrow L^0(=L)\longrightarrow 0$ is an injective resolution of L.

Hence

$$0 \longleftarrow {}^{\sim}Hom(\Gamma_c(L), L) \longleftarrow {}^{\sim}Hom(\Gamma_c(L^0), L^0) \longleftarrow 0$$

is exact. Since A is torsion free

$$0 \longleftarrow {}^{\sim}Hom(\Gamma_c(L), L) \otimes A \longleftarrow {}^{\sim}Hom(\Gamma_c(L^0), L^0) \otimes A \longleftarrow 0$$

is exact. And thus

$$0 \longleftarrow \Gamma_Z({}^{\sim}Hom(\Gamma_c(L), L) \otimes A) \longleftarrow \Gamma_Z({}^{\sim}Hom(\Gamma_c(L^0), L) \otimes A) \longleftarrow 0$$

is exact.

Therefore, $H_0^Z(X:A) \simeq \Gamma_Z({}^{\sim}Hom(\Gamma_c(L), L) \otimes A)$

$$H_{-p}^Z(X:A) = 0, \qquad p = 1, 2, 3, \cdots\cdots$$

For cur proof of i), we shall use the above notations. Since $dim_\phi X < \infty$,

$$H_p^Z(X:A) \simeq H_p(\Gamma_Z(D(I^*) \otimes A).$$

Thus i) can be easily proved by the same way as above.

LEMMA 4. Let z be locally closed in X, z' closed in z and $z'' = z - z'$.

Then there is an exact sequence

$$0 \longrightarrow \Gamma_{Z'}(A) \longrightarrow \Gamma_Z(A) \longrightarrow \Gamma_{Z''}(A)$$

for a sheaf A on X.

Furthermore, if and A is flabby, than

$$0 \longrightarrow \Gamma_{Z'}(A) \longrightarrow \Gamma_Z(A) \longrightarrow \Gamma_{Z''}(A) \longrightarrow 0$$

is exact.

PROOF. $\Gamma_{Z'}(A) = \{s \in \Gamma(A) \mid |s| \subset z'\}$

$V = X - z$ is open in X and $z'' = z \cap V$ is closed in V.

$$\Gamma_{Z''}(A) = \{s \in \Gamma(A|V) \mid |s| \subset z''\}$$

The natural restriction

$$\phi: \Gamma(A) \longrightarrow \Gamma(A|V)$$

induces a map

$$\phi_Z: \Gamma_Z(A) \longrightarrow \Gamma_{Z''}(A)$$

Since $\phi_Z(s)=0$ implies that s is zero in z for $s \in \Gamma_Z(A)$, s has its support in z'. If A is flabby, then ϕ is surjective.

Therefor there is s' in $\Gamma(A)$ restricting to s for $s \in \Gamma_{Z''}(A)$. Since $V=X-z'$ and ϕ is surjective, s' is an extension of element in $\Gamma(A|V)$.

Hence ϕ_Z is surjective.

THEOREM 5. Let z be locally closed in X and z' closed in z and $z''=z-z'$. If a sheaf A is elementary then there is a long exact sequence

$$\cdots \to H^{Z}_{\phi'}(X:A) \longrightarrow H^{Z}_{\phi}(X:A) \longrightarrow H^{Z''}_{\phi}(X:A) \to \cdots$$

PROOF. Since $D(I^*(X:L))$ is flabby and torsion free,

$D_*(I^*(X:L)) \otimes A$ is flabby by lemma 1. By Lemma 4, we have the exact

sequence

$$0 \to \Gamma_{Z'}(D(I^*(X:L)) \otimes A) \to \Gamma_Z(D(I^*(X:L)) \otimes A) \to \Gamma_{Z''}(D(I^*(X:L)) \otimes A \to 0$$

Therefore we have the long exacr sequence.

$$\cdots \to H^{Z'}_{\phi}(X:A) \longrightarrow H^{Z}_{\phi}(X:A) \longrightarrow H^{Z''}_{\phi}(X:A) \to \cdots$$

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