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CURRENT DEVELOPMENTS OF LIE-ADMISSIBLE ALGEBRAS

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1. Algebraic status

For a nonassociative algebra A over a field F, denote by A^- the algebra with multiplication $[x, y]=xy-xy$ defined on the vector space A. Then A is said to be *Lie-admissible* if A^- is a Lie algebra; that is, A^- satisfies the Jacobi identity

$$[[x, y], z]+[[y, z], x]+[[z, x], y]=0. \tag{1}$$

Lie-admissible algebras were first introduced by Albert [1] in 1948 as a counterpart of Jordan-admissible algebras as well as a generalization of Lie algebras. The known study reveals that the algebraic structure of the general Lie-admissible algebras is for from established, at least not at the level of the known structures for algebras such as Lie and Jordan algebras. As indicated in [13], it appears that Lie-admissiblity is alone too general and not of much use in the structure theory. This situation has demanded additional identities in the study of Lie-admissible algebras. In relation to this, the most well-known identity is the *flexible law* $(xy)x=x(yx)$, which was also introduced by Albert [1]. Since the associative, commutative and anticommutative laws satisfy the flexible law, a flexible Lie-admissible algebra is a natural generalization of associative and Lie algebras. Historically, the study of flexible Lie-admissible algebras originated from the Albert's proposal [1] to determine all fiexible algebras A when A^- are semisimple Lie algebras. Laufer and Tomber [7] made the first breakthrough for the Albert's proposal by showing

THEOREM 1. *Let A be a finite-dimensional flexible, power-associative algebra over an algebraically closed field F of characteristic 0. If A^- is a semisimple Lie algebra, then A is itself a Lie algebra isomorphic to A^-.*

In Theorem 1, a crucial assumption on A is power-associativity which forces A to be a nilalgebra [8, 12]. Myung[9] later extended Theorem 1 to a larger class of flexible algebras A over a field of characteristic $\neq 2, 3$ for which A^- is a classical Lie algebra. Very little has been known for the structure of the general flexible Lie-admissible algebras until Okubo[19] has recently constructed the first example of a simple, non-Lie, flexible algebra P_8 over a quadratically closed field of characteristic $\neq 2, 3$ such that $P_8{}^-$ is the 8-dimensional simple Lie algebra su(3). Thus P_8 is neither power-associative nor has a unit element. This algebra P_8, called a ***pseudo-octonion*** or an ***Okubo algebra,*** is constructed on the space of 3×3 trace 0 matrices with a new multiplication given by

$$x*y=\mu xy+(1-\mu)yx-\frac{1}{3}\mathrm{Tr}(xy)I \tag{2}$$

where xy is the matrix product, I is the unit matrix and μ is a fixed scalar subject to $3\mu(1\text{l}-\mu)=1$. The algebra P_8 has a quadratic form which permits the composition law. The construction of P_8 was inspired by the quartic trace identity

$$\mathrm{Tr}x^4=\frac{1}{2}(\mathrm{Tr}x^2)^2 \tag{3}$$

in su(3) which has been studied in particle physics. Okubo [20] subsequently utilized this algebra to construct the octonion algebra (also see [15]). In algebraic point of view, more importantly, the pseudo-octonion algebra laid the groundwork for the following classification by Okubo and Myung[24, 25].

THEOREM 2. *Let A be a finite-dimensional flexible Lie-admissible algebra over an algebraically closed field F of characteristic* 0 *such that A^- is a simple Lie algebra. Then either A is itself a Lie algebra isomorphic to A^-or A^- is a Lie algebra of type $A_n(n\geqslant2)$. In the latter case, A is either a Lie algebra or isomorphic to an algebra with multiplication given by*

$$x*y=\mu xy+(1-\mu)yx-\frac{1}{n+1}\mathrm{Tr}(xy)I \tag{4}$$

which is defined on the space of $(n+1)\times(n+1)$ trace 0 *matrices over F, where xy is the matrix product, $\mu\neq\frac{1}{2}$ is a fixed scalar in F and I is the unit matrix.*

The central idea in Theorem 2 is the notion of adjoint operators in Lie algebras which has been studied in particle physics, notably by Okubo [17, 18]. Specifically, it is shown in [24, 25] that the classification of flexible Lie-admissible algebras A reduces to that of the adjoint operator space $V_0(A^-)$ in the adjoint representation and that if A^- is simple then $\dim V_0(A^-)=2$ for Lie algebras of type $A_n(n\geqslant2)$ and $\dim V_0(A^-)=1$ for all other Lie algebras. This in turn implies that $V_0(A^-)$ contains a nonzero symmetric element only for type $A_n(n\geqslant2)$ and none for all other types. Theorem 2 has been earlier proved for the complex number field by invoking the transcendental method of Wigner-Eckart Theorem [24]. Notice that the pseudo-octonions are special cases of Theorem 2 with $n=2$.

Theorem 2 essentially resolves the Albert's proposal in the characteristic zero case. Theorem 1 is now an immediate consequence of Theorem 2 [25]. Thus the simple algebras associated with type $A_n (n \geqslant 2)$ yield a new family of simple flexible algebras which are neither power-associative nor have a unit element. The classification of these algebras in prime characteristic is a long-standing open problem as it has been even for Lie algebras. Theorem 2 was also utilized to determine all flexible Lie-admissible algebras A with A^- reductive where the Levi-factor of A^- is a simple Lie algebra. During the preparation of this note, I learned that Benkart and Osborn at University of Wisconsin determined all flexible Lie-admissible algebras A when A^- is the direct sum of a Levi-factor and the solvable radical of A^-.

The classification of more general flexible Lie-admissible algebras A has been attempted by means of conditions imposed on the associated Lie algebra A^-. A well known class of such Lie algebras is the class of quasi-classical Lie algebras [24]. A Lie algebra L is called *quasi-classical* if it possesses a nondegenerate symmecric bilinear form (x, y) satisfying the invariant condition

$$([x, y], z) = (x, [y, z]). \tag{5}$$

It is well known that the reductive Lie algebras are quasi-classical [24]. Okubo [23] recently constructed various quasi-calssical Lie algebras which are not reductive. Consequently, there exist flexible Lie-admissible algebras A such that A^- are quasi-classical but not reductive. More generally, the question as to whether one is be able to classify flexible Lie-admissible algebras without reference to the structure of A^- seems to be a quite difficult problem, if not impossible. To support this view, Okubo [23] constructed an infinite class of simple flexible Lie-admissible algebras whose associated Lie algebras are quasi-classical but not reductive. Therefore, the classification of these algebras seems difficult even when their associated Lie algebras are quasi-classical. Nevertheless, the known results indicate that the Cartan theory plays a major role in the study of flexible Lie-admissible algebras and therefore its suructure theory closely resembles the classical theory of Lie algebras (see [9-12]).

On the other hand, non-flexible Lie-admissible algebras are more widely dispersed in its origin and largely arise as byproducts of the study of non-associative algebras defined by identities. For the current status of these algebras, the reader is referred to Myung [13]. However, as we shall see, nonflexible Lie-admissible algebras arise in physics as a methodological tool for the study of dissipative systems in classical Newtonian mechanics as well as in quantum mechanics. Santilli [27, 30, 31, 32] has fiirst shown that such systems can be formulated in terms of a new Poisson bracket which is Lie-admissible.

2. Physical relevences

First we briefly summarize the role of Lie algebras in physics. The significance of Lie algebras in physics is well established on numerous grounds. As the level of Newtonian mechanics Lie algebras emerge as a central methodological tool, Hamilton's equations

$$\dot{a}^{\mu}-\sum_{\nu}\omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}=0,\quad \mu=1,2,\cdots,2n, \tag{6}$$

$$\{a^{\mu}\}=\{\boldsymbol{r},\boldsymbol{p}\},\qquad (\omega^{\mu\nu})=\begin{pmatrix}0&1\\-1&0\end{pmatrix},$$

and more specifically, as the brackets of the time evolution law induced by these equations, i. e.,

$$[A,B]_{\text{classical}}=\sum_{\mu,\nu}\frac{\partial A}{\partial a^{\mu}}\,\omega^{\mu\nu}\,\frac{\partial B}{\partial a^{\nu}}=\sum_{k}\left(\frac{\partial A}{\partial \boldsymbol{r}^{k}}\,\frac{\partial B}{\partial \boldsymbol{p}_{k}}-\frac{\partial B}{\partial \boldsymbol{r}^{k}}\,\frac{\partial A}{\partial \boldsymbol{p}_{k}}\right). \tag{7}$$

In the transition to quantum mechanics, Lie algebras again emerge as a fundamental methodological tool for the characterization of the dynamical evolution of the systems. Indeed, in this case we have the quantum mechanical generalization of Hamilton's equations, that is, the Heisenberg's equations

$$\dot{a}^{\mu}=\frac{1}{i\hbar}[a^{\mu},H], \tag{8}$$

$$[a^{\mu},a^{\nu}]=i\hbar\omega^{\mu\nu},$$

while the brackets of the time evolution law induced by these equations give the Lie product

$$[A,B]=AB-BA, \tag{9}$$

where AB is an associative product.

Recent studies by Santilli [30, 31, 32] and Ktorides [3, 4, 5] have indicated that Lie-admissible algebras have a number of intriguing and significant psysical applications. These can be essentially summarized as follows. Lie algebras have a direct physical significance for Newtonian systems which satisfy the integrability conditions for the existence of action functions, i. e., which are variationally selfadjoint (SA) [28, 29], and we shall write this as

$$\left[m_{k}\ddot{\boldsymbol{r}}_{k}-\boldsymbol{f}_{k}(t,\boldsymbol{r},\dot{\boldsymbol{r}})\right]_{SA}^{C^{\infty}}=0 \tag{10}$$

or equivalently, for the case of forces (or couplings) derivable from a potential as

$$\boldsymbol{f}_{k}=-\frac{\partial V}{\partial \boldsymbol{r}^{k}}+\frac{d}{dt}\frac{\partial V}{\partial \dot{\boldsymbol{r}}^{k}}. \tag{11}$$

Indeed, this property permits the computation of a Hamiltonian $H(t,\boldsymbol{r},\boldsymbol{p})$ in the space of the Cartesian coordinates $\boldsymbol{r}$ of the experimental detection of the system, the linear momentum $\boldsymbol{p}$ and time t. In this case the function H also possesses a direct physical significance (the total energy) and the product $\boldsymbol{M}=\boldsymbol{r}\times\boldsymbol{p}$ is the physical angular mementum. In turn, this canonical formalism allows the direct use of Lie algebras along the lines mentioned above.

The area of direct physical relevance of Lie algebras includes, for instance, the Lorentz force, the electromagnetic interactions, or more generally, the unified gauge theories of weak and electromagnetic interactions.

However, local forces derivable from a potential do not by far exhaust the force of physical reality. This demands the study of forces more general than forces described by (11). The simplest class of nontrivial generalized forces is that of local forces nonderivable from a potential, that is, forces which violate the integrability conditions for the existence of an action function in the assumed local coordinates (nonselfadjoint forces). The studies in [28, 29] have indicated that in this case the construction of a Lagrangian is still possible, $L'=L'(t',\boldsymbol{r}',\dot{\boldsymbol{r}}')$, but it necessarily demands the use of rather complex coordinate transformations, e.g., $t'=t'(t,\boldsymbol{r},\dot{\boldsymbol{r}})$ and $\boldsymbol{r}'=\boldsymbol{r}'(t,\boldsymbol{r},\dot{\boldsymbol{r}})$, with a generally nonlinear dependence on the velocities. This yields the applicability of Lie algebras via the computation of a Hamiltonian $H'(t',\boldsymbol{r}',\boldsymbol{p}')$. However, as a necessary condition for the existence of such a Hamiltonian, the coordinates $\boldsymbol{r}'$ cannot represent those of the experimental detection of the system ($\boldsymbol{r}'$ represents a generally noninertial system owing to the nonlinear dependence on the velocities), the canonical momentum $\boldsymbol{p}'$ cannot represent the physical linear momentum $m\dot{\boldsymbol{r}}$ (indeed, it exhibits an arbitrary functional dependence on $t,\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$), the function H' cannot represent the total mechanical energy, and the product $\boldsymbol{M}'=\boldsymbol{r}'\times\boldsymbol{p}'$ cannot represent the physical angular momentum $(\boldsymbol{r}\times m\dot{\boldsymbol{r}})$.

This situation has created the need of modifying the conventional Hamilton's equations, their time evolution law and consequently the underlying algebraic structure for the case of local forces nonderivable from a potential. This modification should be done in such a way so as to exhibit the direct physical significance of the algorithms. as done for the Lie treatment of the case of selfadjoint forces.

Santilli [31] has described the relevant generalization of Hamilton's equations under the conditions given by

$$\begin{aligned}
&\dot{a}^{\mu}-\sum_{\nu}S^{\mu\nu}(t,a)\frac{\partial H}{\partial a^{\nu}}=0, \qquad \mu=1,\ 2,\cdots\cdots,\ 2n,\\
&\{a^{\mu}\}=\{\boldsymbol{r},\boldsymbol{p}\},\ \ \boldsymbol{p}=m\dot{\boldsymbol{r}}, \qquad\qquad (12)\\
&S^{\mu\nu}=\frac{\partial a^{\mu}}{\partial R_{\nu}},\ \ R=R(t,a),\ \ \det S\neq 0,
\end{aligned}$$

where the tensor $(S^{\mu\nu})$ is the solution of the system

$$\sum_\nu S^{\mu\nu}\frac{\partial H}{\partial a^\nu}=\Xi^\mu, \qquad (\Xi^\mu)=\begin{pmatrix} \boldsymbol{p}/m \\ \boldsymbol{f}+\boldsymbol{F}\end{pmatrix}, \tag{13}$$

$$\boldsymbol{f}=\boldsymbol{f}_{SA}, \quad \boldsymbol{F}=\boldsymbol{F}_{NSA}.$$

The brackets of the generalized time evolution law

$$\dot{A}(a)=\sum_\mu\frac{\partial A}{\partial a^\mu}\dot{a}^\mu=\sum_{\mu,\nu}\frac{\partial A}{\partial a^\mu}\frac{\partial a^\mu}{\partial R_\nu}\frac{\partial H}{\partial a^\nu}=\sum_\nu\frac{\partial A}{\partial R_\nu}\frac{\partial H}{\partial a^\nu} \tag{14}$$

then violate the conditions for a Lie algebra. However they precisely satisfy Lie-admissibility with the bracket

$$A\circ B-B\circ A=\sum_\nu\left(\frac{\partial A}{\partial R_\nu}\frac{\partial B}{\partial a^\nu}-\frac{\partial B}{\partial R_\nu}\frac{\partial A}{\partial a^\nu}\right).$$

The simplest possible case of this is given by the equations

$$\dot{\boldsymbol{r}}=\frac{\partial H}{\partial \boldsymbol{p}}, \quad \dot{\boldsymbol{p}}=-(1+\varepsilon)\frac{\partial H}{\partial \boldsymbol{r}}, \qquad \varepsilon\approx 0$$

which were studied by Duffin [2] for the treatment of systems of oscillators with small damping terms or electric circuits with small internal losses. In this case, as shown by Santilli [27], the brackets of the time evolution law

$$\dot{A}(\boldsymbol{r},\boldsymbol{p})=\frac{\partial A}{\partial \boldsymbol{r}}\cdot\frac{\partial H}{\partial \boldsymbol{p}}-(1+\varepsilon)\frac{\partial A}{\partial \boldsymbol{p}}\cdot\frac{\partial H}{\partial \boldsymbol{r}}$$

are precisely Lie-admissible. Equations (12) are at the foundation of a nontrivial generalization of analytic mechanics whose underlying algebraic structure by construction is strictly non-Lie; however it is Lie-admissible. According to this approach, all algorithms at hand ($\boldsymbol{r},\boldsymbol{p},H,\boldsymbol{M}=\boldsymbol{r}\times\boldsymbol{p}$, etc.) possess a direct physical significance (which would be precluded by a conventional canonical approach), despite the fact that the forces are nonderivable from a potential. Also, the conventional analytic mechanics is recovered identically at the limit of null forces nonderivable from a potential.

In the same vein, Santilli [32] has also indicated that Lie-admissible algebras emerge for the quantum mechanical treatment of nonselfadjoint forces. In this case he has worked out the following generalization of Heisenberg's equations

$$\dot{a}^\mu=\frac{1}{i\hbar}(\dot{a}^\mu,H), \quad (a^\mu,a^\nu)=i\hbar S^{\mu\nu}$$

with underlying product

$$(A,B)=ARB-BSA, \quad R\neq\pm S, \quad R,S, \text{ nonsingular} \tag{15}$$

where ARB indicates an associative product and A,B are operators in a Hilbert space. It can be shown that an algebra with multiplication given by (15) is Lie-admissible but not flexible in geneal (Myung [13, section 7]). These more general algebras emerge as fundamental mathematical tools for the characterization of the dynamical evolution of the more general systems considered in the same way as Lie algebras emerge for the case of selfadjoint forces. To begin the sutdy of these generalized formulations with a special case, Santilli [26] introduced the following product

$$(A, B) = \lambda AB - \mu BA, \quad \lambda, \mu \in F \tag{16}$$

in an associative algebra which gives rise to a flexible Lie-admissible algebra. Clearly the product (16) is a particular case of the product (15) when the operators R and S are chosen from the field. This flexible Lie-admissible algebra has been called the (λ, μ)-mutation algebra of an associative algebra.

The (λ, μ)-mutation algebras are currently used in a number of physical applications, such as the construction of the Gell-Mann-Okubo mass formula by Ktorides [3], or the proposal by Santili [32] and Ktorides, Myung and Santilli [6] for the experimental verification as to whether Pauli's exclusion principle is strictly valid in nuclear physics or whether very small deviations are detectable. More recently, Myung and Santilli [16] investigated a more generalized mutation arising from (15). If $S = \lambda R$ in (15), then the product defined by (15) gives rise to a flexible Lie-admissible algbera, called the $(R, \lambda R)$-mutation. It is pointed out in [16] that the $(R, \lambda R)$-mutation with $\lambda \neq \pm 1$ applied to an enveloping algebra of the su(2), spin $\frac{1}{2}$, Lie algebra provides a hierarchy of breakings of the su(2)-spin symmetry and a Lie-admissible covering of the conventional Lie notion of spin, along a recent conjecture by Santilli.

Perhaps, the most potentially significant application of Lie-admissible algebras which has recently emerged is the fundamental open problem of contemporary physics, namely, that of the structure of the strongly interacting particles(hadrons). In essence, recent studies initiated by Santilli [32] and now under investigation by a number of authors have indicated the possible physical origin of the theory that the strong interactions in general and the strong hadronic forces in particular demand forces more general than the atomic and nuclear forces, that is, they demand local nonselfadjoint forces. This situation makes Lie-admissible algebras directly applicable to hadron structure. As a matter of fact, their application plays a crucial role fully parallel, although of generalized nature, to that of Lie algebras for atomic structure. Indeed, Lie-admissible algebras first emerge for the characterization of the dynamical evolution of the hadronic constituents under these broader forces

$$\dot{A} = \frac{1}{i\hbar}\,(ARH - HSA)$$

and, secondly, they emerge for the characterization of a generalization of the Galilei relativity of the type

$$\hat{A}'=e^{-\frac{\theta_i}{i\hbar}X_iR}\ Ae^{\frac{\theta_j}{i\hbar}SX_j}$$

where the X's(θ's) are the conventional generators (parameters) of the Galilei group.

According to this view, when the hadronic constituents are approximated with point-like particles, their strong forces are derivable from a potential and Lie algebras emerge as applicable as in conventional conservative mechanics. Instead, when the hadronic constituents are assumed as being extended charged particles, their interactions at distances smaller than their charge diameter implies generalized forces nonderivable from a potential. This makes Lie-admissible algebras directly applicable along the quantum mechanical lines noted above. In turn, this implies a mutation of the notion of particle in the transition from electromagnetic interactions to the strong interactions. For instance, the su(2)-spin algebra

$$A'=e^{-\frac{\theta_i}{i\hbar}M_i}\cdot Ae^{\frac{\theta_j}{i\hbar}M_j}$$

(where the θ's are the Euler's angles) is broken by nonselfadjoint strong interactions, resulting in the covering su(2)-admissible structure

$$\hat{A}'=e^{-\frac{\theta_i}{i\hbar}M_iR}\ Ae^{\frac{\theta_j}{i\hbar}SM_j}.$$

Next we turn to Okubo's introduction of flexible Lie-admissible algebres into the quantum mechanics. The central idea in Okubo's approach stems form the fact that an algebra A is flexible Lie-admissible if and only if A satisfies

$$[xy, z]=x[y, z]+[x, z]y \tag{17}$$

for all $x, y, z\in A$ [9]. In other words, if A is flexible Lie-admissible then, for each $x\in A$, the adjoint mapping ad $x : y\to$ad x $(y)=[x, y]$ is a derivation of A. Let A be a nonassociative algebra of operators in the physical system. Then the most general time development of any element $x\in A$ will be expressed as

$$\frac{dx}{dt}=F(x), \tag{18}$$

where F is a function of A into itself. But, in view of (17), the condition of an algebra demands that F is a derivation of A; that is, $F(xy)=F(x)y+xF(y)$, $x, y\in A$. Therefore, if A is a flexible Lie-admissible algebra and all derivations of A^- are inner then (17) implies that

$$F(x)=i[H, x],\quad iH\in A, \tag{19}$$

so that (18) is now written as

$$\frac{dx}{dt}=i[H, x] \tag{20}$$

which one recognizes as the Heisenberg's equation, although the underlying algebra need not be associative but flexible Lie-admissible. The self-consistent quantization requires some additional conditions such as the relation (17) as well as the canonical commutation relation. Since (17) is equivalent to the algebra A being flexible Lie-admissible, Okubo [22] argues that the most reasonable generalization of the present quantum mechanical framework into nonassociative algebras is by means of the flexible Lie-admissible algebras. However, some additional conditions may be necessarily imposed on the flexible Lie-admissible algebra A. For instance, if one also wants to develop the Schrödinger formulation of the quantum mechanics with the state vector Ψ satisfying

$$i\frac{d\Psi}{dt}=H\Psi \tag{21}$$

then we may require the existence of the time-development operator

$$e^{iHt}\equiv\exp(iHt). \tag{22}$$

But, in a flexible Lie-admissible algebra, the exponential function is not in general well-defined unless H is power-associative in the sense that the subalgebra generated by H is associative and so $H^mH^n=H^{m+n}$ for integers $m, n\geq 0$. If H is power-associative, then one can define the exponential function at least formally by

$$e^{iHt}=\sum_{n=0}^{\infty}\frac{1}{n!}(iHt)^n. \tag{23}$$

Furthermore, if H satisfies the relation

$$(H^mx)H^n=H^m(xH^n) \tag{24}$$

for all integers $m, n\geq 0$ and all $x\in A$, then the expression

$$e^{iHt}xe^{-iHt}$$

is also well-defined in view of (23). In this case, any solution of the Heisenberg equation (20) is written as

$$x=x(t)=e^{iHt}x(0)e^{-iHt} \tag{25}$$

just as in the conventional quantum mechanics.

The pseudo-octonion P_8 described by (2) offers an important example of a flexible Lie-admissible algebra which possesses a power-associative element satisfying (24). Let

$$(x,\ y)=\frac{1}{6}\mathrm{Tr}\,xy \tag{26}$$

for $x, y\in P_8$. Then it is shown in [19] that (x, y) is a nondegenerate symmetric bilinear form which satisfies the composition law and the invariant condition

$$\begin{aligned}(x*y,\ x*y)&=(x, x)(y, y)\\ (x*y,\ z)&=(x,\ y*z),\end{aligned} \tag{27}$$

as well as the relation

$$x*(y*z)+z*(y*x)=(x*y)*z+(z*y)*x=2(x, y)z+2(y, z)x. \tag{28}$$

If one chooses the Gell-Mann's eight trace 0 hermitian matrices

$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\quad \lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$

$$\lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},\quad \lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$\lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \lambda_8=\begin{pmatrix}\frac{1}{\sqrt{3}}&0&0\\0&\frac{1}{\sqrt{3}}&0\\0&0&-\frac{2}{\sqrt{3}}\end{pmatrix}$$

then the λ_i form a basis for P_8 and have the multiplication table

$$\lambda_i*\lambda_j=\sum_{k=1}^{8}\left(d_{ijk}\pm\frac{1}{\sqrt{3}}f_{ijk}\right)\lambda_k, \tag{29}$$

together with the relation

$$(\lambda_i,\lambda_j)=\frac{1}{3}\delta_{ij}. \tag{30}$$

If we set

$$h=\sqrt{3}\lambda_8 \tag{31}$$

and write $h*h$, for brevity, as $hh=h^2$, then we find

$$hh=-h,\quad (h,h)=1. \tag{32}$$

In particular, h is power-associative and

$$h^n=\begin{cases}h & \text{if } n \text{ is odd } (n\geqslant 1)\\ -h & \text{if } n \text{ is even } (n\geqslant 2).\end{cases} \tag{33}$$

Since P_8 is flexible, in view of (33) h clearly satisfies (24). Therefore, if we choose the Hamiltonian H as $H=\beta h$, $\beta\in F$ then the Heisenberg equation (20) of motion has a solution of the form

$$\begin{aligned}x(t)&=e^{iHt}x_0e^{-iHt}, \qquad x_0=x(0)\\ &=5x_0-2\{e^{i\beta t}hx_0+e^{-i\beta t}x_0h\}\end{aligned} \tag{34}$$

using (33).

More recently. Okubo [23] gave another argument that the use of non-associative algebras, preferably flexible Lie-admissible algebras, may be of some help for resolving the difficult problem of the color confinement in Quantum-Chromo-Dynamics.

3. Concluding remarks

As is well-known, when Lie algebras made their appearance in physics, their structure was already well developed as a result of the work of Killing, S. Lie, E. Cartan and others. This mathematical development proved to be invaluable for the achievement of physical results. At this time, when Lie-admissible algebras are making their appearance in physics, its structure theory is very little known in comparison to the development of Lie algebras at the time of their application. However, recent studies indicate that the relevance of Lie-admissible algebras in physics, especially in hadron physics, is rapidly growing and becoming more extensive, and it is no exaggeration that the recent activities in Lie-admissible algebras were overwhelmed by the influence from physics and theoretical mechanics.

Since Lie-admissible algebras have some relation to the general theory of nonassociative algebras which has required an immense effort by a number of authors during the past theree decades, this note is by no means exhaustive and has purposely omitted some results which also have relevance to physics in different aspects. For this, the reader may find appropriate references from Okubo [22, 23] and Santilli [30-32].

In preparing this note, I am greatly indebted to two physicists, professors Okubo and Santilli, who patiently assisted and encouraged me to attempt this writing, in spite of my nebulous knowledge in physical areas. Indeed, I must admit that the second part of Secton 2 is largely taken from the recent work of Okubo [22], while the first part is drawn from written and verbal communications with Santilli. However, almost needless to say, I am solely responsible for the content of this note.

It is hoped that this note might inspire young mathematicans to persue this new area which has a far reaching mathematial potential.

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