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ON LIE-ADMISSIBLE ALGEBRAS ASSOCIATED WITH INVARIANT BILINEAR FORMS

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1. Introduction

For a nonassociative algebra A, denote by A^- the algebra with multiplication $[x, y]=xy-yx$ defined on the vector space A. Then A is said to be *Lie-admissible* if A^- is a Lie algebra; that is, A^- satisfies the Jacobi identity

$$[[x, y], z]+[[y, z], x]+[[z, x], y]=0. \tag{1}$$

The well-known examples of Lie-admissible algebras are the associative algebras and Lie algebras, and various types of Lie-admissible algebras have recently been studied in conjunction with physical applications. In fact, Lie-admissible algebras arise both in the classical and quantum mechanics. For this, the reader may be referred to Santilli [10], Okubo [8] and Myung [6].

The objective of this note is to construct various types of Lie-admissible algebras from an algebra with an invariant bilinear form. Our construction is motivated by two forms of mutations which were investigated in the recent work of Ktorides, Myung and Santilli [3] and Okubo and Myung [9]. Specifically, let A be an associative algebra over a field F with product xy. For fixed scalars λ, μ with $\lambda\neq\mu$ in F, we denote by $A(\lambda, \mu)$ the algebra with multiplication

$$x*y=\lambda xy+\mu yx \tag{2}$$

defined on the vector space A. The algebra $A(\lambda, \mu)$ is called the (λ, μ)*-mutation* of A. Another type of mutation stems from the recent classification of simple flexible Lie-admissible algebras given by Okubo and Myung [9]. Let M_n be the vector space of $(n+1)\times(n+1)$ trace 0 matrices over a field F of characteristic 0. For a fixed scalar $\mu\neq\frac{1}{2}$, define a new multiplication on M_n as

$$x*y=\mu xy+(1-\mu)yx-\frac{1}{n+1}\operatorname{Tr}(xy)I \tag{3}$$

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where xy is the matrix product and I is the $(n+1)\times(n+1)$ unit matrix. In fact, it is shown in [9] that if A is a flexible algebra over an algebraically closed field of characteristic 0 such that A^- is a simple Lie algebra, then A is either a Lie algebra or an algebra given by (3). The latter case occurs only when A^- is a simple Lie algebra of type A_n $(n\geq 2)$.

Our construction is designed to generalize both mutations described by (2) and (3). Finally, we relate the present construction to quasi-associative algebras.

2. The construction

Let A be a nonassociative algebra over a field F. A symmetric bilinear form (,) defined on A is called *invariant* if it satisfies

$$(xy, z)=(x, yz) \tag{4}$$

for all $x, y, z\in A$. Also, a linear functional t on A is called an *invariant form* on A if it satisfies

$$t(xy)=t(yx), \tag{5}$$

$$t((xy)z)=t(x(yz)) \tag{6}$$

for $x, y, z\in A$. If t is an invariant form on A, then setting

$$(x, y)=t(xy)$$

defines an invariant bilinear form on A. Suppose now that A has a unit element e, i.e., $xe=ex=x$ for $x\in A$. If (,) is an invariant bilinear form on A then setting $(e, x)\equiv t(x)$ gives an invariant form t on A.

There are two well-konwn classes of algebras which possess an invariant bilinear form. If L is a Lie algebra over F and ρ is a finite-dimensional representation of L, then the trace form

$$(x, y)\equiv \mathrm{Tr}\rho(x)\rho(y) \tag{7}$$

and, in particular, the Killing form are invariant bilinear forms on L. An algebra A over F is called a *noncommutative Jordan algebra* if it satisfies the flexible law and Jordan identity

$$(xy)x=x(yx), \tag{8}$$

$$(x^2y)x=x^2(yx). \tag{9}$$

Denote by R_x and L_x the right and left multiplications by x in A; i.e., $yR_x=yx$ and $yL_x=xy$. Schafer [11] has shown that if A is a finite-dimensional noncommutative Jordan algebra over a field of characteristic 0 then

$$(x, y)\equiv\frac{1}{2}\mathrm{Tr}(R_{xy}+L_{xy}) \tag{10}$$

becomes an invariant bilinear form on A. Notice also that the associative algebras and the Lie algebras are noncommutative Jordan algebras.

Let A now be any algebra over F with an invariant bilinear form (,).

Let c be a fixed element in A. For two fixed scalars λ, μ with $\lambda \neq \mu$ in F, we introduce a multiplication in the vector space A by

$$x*y=\lambda xy+\mu yx-(x, y)c \tag{11}$$

and denote the resulting algebra by $A(\lambda, \mu, c)$. To facilitate the computation, denote the associator and commutator in $A(\lambda, \mu, c)$ by

$$(x, y, z)^*=(x*y)*z-x*(y*z), \tag{12}$$

$$[x, y]^*=x*y-y*x. \tag{13}$$

We compute from (11)

$$\begin{aligned}(x*y)*z=&\lambda[\lambda(xy)z+\mu z(xy)-(xy, z)c]+\mu[\lambda(yx)z+\mu z(yx)-(yx, z)c]\\&+(x, y)[\lambda cz+\mu zc-(c, z)c],\\ x*(y*z)=&\lambda[\lambda x(yz)+\mu(yz)x-(x, yz)c]+\mu[\lambda x(zy)+\mu(zy)x-(x, zy)c]\\&-(y, z)[\lambda xc+\mu cx-(x, c)c]\end{aligned}$$

and consequently

$$\begin{aligned}(x, y, z)^*=&\lambda^2(x, y, z)-\mu^2(x, y, z)+\lambda\mu[z(xy)+(yx)z-(yz)x-x(zy)]\\&+\lambda[(y, z)xc-(x, y)cz]+\mu[(y, z)cx-(x, y)zc]\\&+[(x, y)(c, z)-(y, z)(x, c)]c\end{aligned} \tag{14}$$

where (x, y, z) and $[x, y]$ respectively indicate the associator and commutator in A. In particular, (14) implies

$$(x, y, x)^*=(\lambda^2-\mu^2)(x, y, x)+(\lambda-\mu)(x, y)[x, c]. \tag{15}$$

Therefore, if A is flexible and $[A, c]=0$ then $A(\lambda, \mu, c)$ is flexible. Also we note

$$[x, y]^*=(\lambda-\mu)[x, y] \tag{16}$$

and hence $A(\lambda, \mu, c)^-$ is isomorphic to A^- via the mapping $x \rightarrow \frac{1}{\lambda-\mu}x, x \in A$. This shows that $A(\lambda, \mu, c)$ is Lie-admissible if and only if A is Lie-admissible. We summarize these in

THEOREM 1. *Let A be an algebra over a field F with an invariant bilinear form. For a fixed element $c \in A$ and λ, μ with $\lambda \neq \mu$ in F, let $A(\lambda, \mu, c)$ be the algebra defined by* (11). *Then $A(\lambda, \mu, c)$ is Lie-admissible if and only if A is. Furthermore, if A is flexible and $[c, A]=0$ then $A(\lambda, \mu, c)$ is flexible also.*

If A has an invariant linear form t, then we define a multiplication in A by

$$x*y=\lambda xy+\mu yx-t(xy)c. \tag{17}$$

When (x, y) is replaced by $t(xy)$, the relations (14)-(16) with (17) hold without modification. If one takes $c=0$ then $A(\lambda, \mu, 0)$ is the (λ, μ)-mutation of A given by (2). On the other hand, if A is the $(n+1)\times(n+1)$ matrix algebra over F and we set $(x, y)=\frac{1}{n+1}\mathrm{Tr}(xy)$, then $A(\lambda, 1-\lambda, 1)$ is the algebra described by (3). Therefore, the present construction generalizes two mutations given by (2) and (3) both in its setting and underlying algebraic structure of the given algebra.

The underlying algebraic structure of $A(\lambda, \mu, c)$ is quite different from that of A while A^- and $A(\lambda, \mu, c)^-$ have the same structure. For example, when A is flexible or even associative, $A(\lambda, \mu, c)$ is not in general flexible. A well-known example for this is the pseudo-octonion algebra introduced by Okubo [7]. Here we give another example for the situation. Let L be a finite-dimensional semisimple Lie algebra over a field F of characteristic 0 and let (,) be the Killing form. Then by (15), $L(\lambda, \mu, c)$ is flexible if and only if $(x, y)[x, c]=0$ for all $x, y \in L$. Since the center of L is 0, there exists an $x \neq 0$ in L such that $[x, c] \neq 0$, Thus if $L(\lambda, u, c)$ is flexible, one would have $(x, y)=0$ for all $y \in L$ and this is absurd, since (,) is nondegenerate. Thus $L(\lambda, \mu, c)$ for $c \neq 0$ can not be flexible.

The algebras in the above construction are all finite-dimensional. However, an algebra of infinite dimension may be constructed as follows. Let L be a finite-dimensional Lie algebra over a field F of characteristic 0 and let ρ be a finite-dimensional representation of L acting on a vector space V over F. Let $U(L)$ be the universal enveloping algebra of L. Then ρ is extended to a unique homomorphism of $U(L)$ into $Hom_F V$. Therefore, we can define a trace form on $U(L)$ by

$$t_\rho(x)=\mathrm{Tr}\rho(x), \quad x \in U(L).$$

Then the bilinear form (,) on $U(L)$ defined by $(x, y)_\rho=t_\rho(xy)$ is clearly invariant. Thus the algebra $U(L)(\lambda, \mu, c)$ is defined by means of (11) and is flexible Lie-admissible if c is in the center of $U(L)$; for example, if c is a Casimir invariant.

3. Quasi-associative algebras

An important subclass of noncommutative Jordan algebras is the class of quasi-associative algebras. An algebra A over a field F is called *quasi-associative* if there exists an extension field K of F and an associative algebra B over K such that $A_K=B(\lambda)$ for some $\lambda \in K$, where A_K is the scalar extension of A to K and $B(\lambda)$ is the algebra with multiplication

$$x \circ y=\lambda xy+(1-\lambda)yx. \tag{18}$$

Any quasi-associative algebra is flexible Lie-admissible [1]. Let B be an associative algebra over F and let p be an element in B. For a fixed scalar $\lambda \neq -1$ in F, define a product in B by

$$x*y=xpy-\lambda ypx \tag{19}$$

and denote the resulting algebra by $B(p, \lambda p)$, called the $(p, \lambda p)$-mutation of

B. Then it is easily seen that $B(p,\lambda p)$ is flexible Lie-admissible.

Also, we have $x*y=\mu x(\alpha p)y+(1-\mu)y(\alpha p)x$ where $\mu=\frac{1}{1-\lambda}$ and $\alpha=1-\lambda$. If we donte by $B^{(\alpha p)}$ the algebra with multiplication $x\cdot y=x(\alpha p)y$ then $B(p,\lambda p)=B^{(\alpha p)}(\mu)$. Since $B^{(\alpha p)}$ is associative, $B(p,\lambda p)$ is quasi-associative. Notice also that the (λ,μ)-mutation is a special case of the $(p,\lambda p)$-mutation. The algebra $B(p,\lambda p)$ has been investigated in [5] and arises from a generalization of the Heisenberg equation in the quantum mechanics [10].

It follows from (19) that $B(p,\lambda p)$ satisfies the identity

$$(x,y,z)^*=\frac{1}{(1+\lambda)^2}[[x,z]^*,y]^*. \tag{20}$$

In this section, we shall show that (20) is a necessary and sufficient condition for an algebra to be quasi-associative. For this, we need the following theroem proved by Albert [1].

THEOREM 2. *Let A be an algebra over a field F of characteristic $\neq 2$ which is neither associative nor a Jordan algebra. Then A is quasi-associative if and only if A is flexible and there exists a scalar $\alpha_0\neq 0,-\frac{1}{4}$ in F such that*

$$-(2\alpha_0+1)(x,y,z)=\alpha_0[x(zy)+(yz)x-(yx)z-z(xy)] \tag{21}$$

holds for all $x,y,z\in A$.

In an algebra A, we introduce the notation

$$S(x,y,z)\equiv(x,y,z)+(y,z,x)+(z,x,y).$$

If A satisfies $(x,x,x)=0$ for all $x\in A$ then it is shown in [2] that A also satisfies the identity

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=2S(x,y,z). \tag{22}$$

The general form of (20) in any algebra A is written as

$$(x,y,z)=\alpha[[z,x],y],\quad \alpha\in F. \tag{23}$$

LEMMA 3. *Let A be an algebra over F of characteristic $\neq 2$. If A satisfies (23) with $\alpha\neq\frac{1}{2}$ then A is noncommutative Jordan and Lie-admissible.*

Proof. Setting $z=x$ and $z=x^2$ in (23) gives in turn the flexible law $(x,y,x)=0$ and the Jordan identity $(x^2,y,x)=0$. By permuting $x\to y\to z\to x$, we obtain from (23)

$$\alpha[[y,z],x]=(z,x,y),\quad \alpha[[x,y],z]=(y,z,x)$$

and adding these to (23) gives

$$2\alpha S(x,y,z)=S(x,y,z) \tag{24}$$

in view of (22). Since $\alpha\neq\frac{1}{2}$, this implies that $S(x,y,z)=0$ and hence A is Lie-admissible by (22).

THEOREM 4. *Let A be an algebra over F of characteristic $\neq 2$. If A is flexible Lie-admissible then* (21) *implies* (23) *with $\alpha_0=\alpha$. If A satisfies* (23) *with $\alpha\neq\frac{1}{2}$ then A is quasi-associative.*

Proof. Suppose that A is a flexible Lie-admissible algebra satisfying (21). Then the right side of (21) becomes

$$\begin{aligned}&x(zy)+(yz)x-(yx)z-z(xy)\\&=[[x,z],y]-(xz)y+(zx)y\\&\qquad+y(xz)-y(zx)+x(zy)+(yz)x-(yx)z-z(xy)\\&=[[x,z],y]-(x,z,y)+(z,x,y)-(y,x,z)+(y,z,x)\\&=[[x,z],y]+2(y,z,x)+2(z,x,y)\end{aligned}$$

by the flexible law. Thus (21) is written as

$$-(2\alpha_0+1)(x,y,z)=\alpha_0\{[[x,z],y]-2(x,y,z)\} \tag{25}$$

since A is Lie-admissible and so $S(x,y,z)=0$ by (22). Thus (25) is nothing but (21) with $\alpha=\alpha_0$. Suppose now that A satisfies (23) with $\alpha\neq\frac{1}{2}$, Then by Lemma 3, A is flexible Lie-admissible and $S(x,y,z)=0$ in A. Thus (23) becomes

$$-(x,y,z)=\alpha\{[[x,z],y]+2S(x,y,z)\}$$

and expanding this gives

$$\begin{aligned}-(2\alpha+1)(x,y,z)&=\alpha\{[[x,z],y]+2(y,z,x)+2(z,x,y)\}\\&=\alpha[x(zy)+(yz)x-(yx)z-z(xy)],\end{aligned}$$

which is (21).

As an immediate consequence of Theorems 2 and 4, we have

COROLLARY 5. *Let A be an algebra over F of characteristic $\neq 2$ which is neither associative nor Jordan. Then A is quasi-associative if and only if A satisfies* (23) *with $\alpha\neq\frac{1}{2}$.*

Let $A(\lambda,\mu)$ be the (λ,μ)-mutation of a noncommutative Jordan algebra A over F of characteristic 0. Denote by R_x^* and L_x^* the right and left multiplications by x in $A(\lambda,\mu)$. Then we have

$$R_x^*=\lambda R_x+\mu L_x,\quad L_x^*=\lambda L_x+\mu R_x \tag{26}$$

and hence

$$t^*(x)\equiv\frac{1}{2}\mathrm{Tr}(L_x^*+R_x^*)=\frac{1}{2}(\lambda+\mu)\mathrm{Tr}(L_x+R_x). \tag{27}$$

Since $\frac{1}{2}\mathrm{Tr}(L_x+R_x)$ is an invariant form on A, so is t^* on $A(\lambda,\mu)$. This in particular applies to an associative algebra A.

4. The kernel of an invariant form

In order to relate the present construction more closely to one described by (3), let A be a finite dimensional algebra over a field F of characteristic 0 with an invariant form t and let A have a unit element e. If t is not trivial then $t(c)\neq 0$ and hence A is composed as $A=Fe\oplus A_0$ where A_0 is the kernel of t. Notice that A_0 is not in general a subalgebra of the algebra $A(\lambda,\mu,e)$ constructed by (11) though A_0 is always a subalgebra of A^- or $A(\lambda,\mu,e)^-$. However, if one choose λ,μ so as to satisfy $\lambda+\mu-f(e)=0$, then A_0 is a subalgebra of $A(\lambda\mu,e)$ since

$$t(x*y)=\lambda t(xy)+\mu t(yx)-t(xy)f(e)=(\lambda+\mu-f(e))t(xy)=0.$$

By dividing by $f(e)$, we can further assume that $f(e)=1$ and thus $\mu=1-\lambda$. Therefore,

A_0 becomes an algebra over F with multiplication

$$x*y=\lambda xy+(1-\mu)yx-(x,y)e \tag{28}$$

where $(x,y)=t(xy)$, When A is the $(n+1)\times(n+1)$ matrix algebra over F and $(x,y)=\frac{1}{n+1}\mathrm{Tr}(xy)$, (28) gives (3) as a special case.

In the remainder of this section, we investigte a method to recover the original algebra A as a mutation of A_0. Assume $\lambda\neq\frac{1}{2}$ Firse we take the vector space direct sum $A_0\oplus Fe$ and define a multiplication in $A_0\oplus Fe$ as

$$(x+\alpha e)\circ(y+\beta e)=(x*y+\alpha y+\beta x)+(\alpha\beta+(x,y))e \tag{29}$$

where $x,y\in A_0$, $\alpha,\beta\in F$ and $x*y$ is given by (28). Then (29) is rewritten as

$$\begin{aligned}(x+\alpha e)\circ(y+\beta e)&=[\lambda xy+(1-\lambda)yx+\alpha y+\beta x]+\alpha\beta e\\&=\lambda(x+\alpha e)(y+\beta e)+(1-\lambda)(y+\beta e)(x+\alpha e).\end{aligned}$$

Therefore, $A_0\oplus Fe=A(\lambda)$, the $(\lambda,1-\lambda)$-mutation of A, and thus $A=(A_0\oplus Fe)(\mu)$ with $\mu=(2\lambda-1)^{-1}\lambda$. Okubo [8] has proved this when A is the $(n+1)\times(n+1)$ matrix algebra. Since quasi-associativity is transitive [1], we have proved.

THEOREM 6. *Let A be an algebra over a field F of characteristic 0 with an invariant bilinear form and let A have a unit element e. Then A is quasi-associative if and only if the algebra $A_0\oplus Fe$ given by (29) is quasi-associative.*

If A is the quasi-associative algebra in Theorem 6 then, by (28) and (14), we have

$$\begin{aligned}(x,y,z)^*=\lambda^2(x,y,z)+(1-\lambda)^2(x,y,z)+\lambda(1-\lambda)[z(xy)+\\(yx)z-(yz)x-x(zy)]+(y,z)x-(x,y)z,\end{aligned}$$

since A is flexible and e is the unit element. In view of Theorm 2, this reduces to

$$(x, y, z)*=\gamma(x, y, z)+(y, z)x-(x, y)z \tag{30}$$

where $\gamma=\lambda^2+(1-\lambda)^2+\lambda(1-\lambda)(2+\frac{1}{\alpha})$ for some $\alpha\neq 0, \frac{1}{2}$. Okubo [8] has obtained an identity similar to (30) for an associative algebra.

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