Bull. Korean Math. Soc.
Vol. 16, No. 2, 1980

ON SOME RELATIONS OF TWO 2-DIMENSIONAL UNIFIED FIELD THEORIES

BY PHIL UNG CHUNG

1. Introduction

A. **2-dimensional *g-UFT* and **g-UFT*.** In the usual Einstein's unified field theory (g-UFT) the generalized 2-dimensional Riemannian space X_2 referred to a real coordinate system x^ν is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$(*):

(1.1)a $$g_{\lambda\mu}=h_{\lambda\mu}+k_{\lambda\mu},$$

where

(1.1)b $$\mathfrak{g}=\mathrm{Det}(g_{\lambda\mu})\neq 0,\quad \mathfrak{h}=\mathrm{Det}(h_{\lambda\mu})\neq 0,\quad \mathfrak{K}=\mathrm{Det}(k_{\lambda\mu})\neq 0.$$

The tensor $h_{\lambda\mu}$ together with $h^{\lambda\nu}$, uniquely defined by

(1.2) $$h_{\lambda\mu}h^{\lambda\nu}=\delta_\mu^\nu,$$

are used for raising and/or lowering indices in 2-dimensional g-UFT.

The differential geometric structure is imposed on X_2 by the tensor $g_{\lambda\mu}$ by means of a connection $\Gamma_\lambda{}^\nu{}_\mu$ given by the system of Einstein's equations [3]

(1.3) $$D_w g_{\lambda\mu}=2S_{w\mu}{}^\alpha g_{\lambda\alpha},$$

where D_w denotes the symbol of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$, and $S_{\lambda\mu}{}^\nu=\Gamma^\nu_{[\lambda\mu]}$.

On the other hand, 2-dimensional $*g^{\lambda\nu}$-unified field theory ($*g$-UFT) in the same space X_2 referred to a real coordinate system x^ν is defined to be based upon the real nonsymmetric tensor $*g^{\lambda\nu}$ defined by

(1.5) $$g_{\lambda\mu}{}^*g^{\lambda\nu}=\delta_\mu^\nu.$$

It may also be decomposed into its symmetric part $*h^{\lambda\nu}$ and skew-symmetric part $*k^{\lambda\nu}$:

(1.6) $$*g^{\lambda\nu}=*h^{\lambda\nu}+*k^{\lambda\nu}.$$

Since $\mathrm{Det}(*h^{\lambda\nu})\neq 0$, we may define the tensor $*h_{\lambda\mu}$ by

(1.7) $$*h_{\lambda\mu}{}^*h^{\lambda\nu}=\delta_\mu^\nu.$$

*In the 2-dimensional *g-UFT we use both $*h_{\lambda\mu}$ and $*h^{\lambda\nu}$ as tensors for raising and/or lowering indices of all starred tensors defined in X_2 in the usual manner.*

(*) Throughout the present paper, Greek indices take the values 1, 2 and follow the summation convention.

We then have, for example,

(1.8)a $\quad {}^*k_{\lambda\mu}={}^*k^{\rho\sigma}\,{}^*h_{\lambda\rho}\,{}^*h_{\mu\sigma},\quad {}^*g_{\lambda\mu}={}^*g^{\rho\sigma}\,{}^*h_{\lambda\rho}\,{}^*h_{\mu\sigma},$

so that

(1.8)b $\quad {}^*g_{\lambda\mu}={}^*h_{\lambda\mu}+{}^*k_{\lambda\mu}.$

Similarly the differential geometric structure in 2-dimensional *g-UFT is imposed on X_2 by means of a connection $\Gamma^{\nu}_{\lambda\mu}$ given by the following system of equations equivalent to (1.3):

(1.9) $\quad D_{\omega}{}^*g^{\lambda\nu}=-2S_{\omega\alpha}{}^*g^{\nu\lambda\alpha}.$

Using the following densisties andascalars, we define the following:

(1.10)
$$\begin{aligned}&{}^*\mathcal{G}=\mathrm{Det}({}^*g_{\lambda\mu}),\quad {}^*\mathfrak{h}=\mathrm{Det}({}^*h_{\lambda\mu}),\\ &{}^*\mathcal{K}=\mathrm{Det}({}^*k_{\lambda\mu}),\quad g=\mathcal{G}/\mathfrak{h},\\ &k=\mathcal{K}/\mathfrak{h},\quad {}^*g={}^*\mathcal{G}/{}^*\mathfrak{h},\quad {}^*k={}^*\mathcal{K}/\mathfrak{h},\end{aligned}$$

Chung [1] proved that two unified tensor fields $g_{\lambda\mu}$ and ${}^*g_{\lambda\mu}$ are related by

(1.11)a $\quad {}^*h^{\lambda\nu}=\frac{1}{g}h^{\lambda\nu},\quad {}^*k^{\lambda\nu}=\frac{1}{g}k^{\lambda\nu},$

(1.11)b $\quad {}^*h_{\lambda\mu}=gh_{\lambda\mu},\quad {}^*k_{\lambda\mu}=gk_{\lambda\mu},$

and that

(1.12) $\quad g=1+k,\quad {}^*g=1+{}^*k.$

In both 2-dimensional unified field theories, it is obvious that *there exists only the first class* since

$$\mathcal{K}=(k_{12})^2>0,\quad {}^*\mathcal{K}={}^*(k_{12})^2>0.$$

B. **Purpose.** The purpose of the present paper is to derive some relations of 2-dimensional g-UFT and *g-UFT other than (1.11) and (1.12). These results are used to investigate the relationship between two different expressions of torsion tensor $S_{\lambda\mu}{}^{\nu}$ which lead to a solution of (1.3) and (1.9) in Einstein's 2-dimensional unified field theories.

2. Some relations of 2-dimensional g-UFT and *g-UFT

In this Section, we derive several relations of two 2-dimensional unified field theories and obtain a simple expression for the torison tensor in 2-dimensional *g-UFT.

THEOREM (2.1). *The scalars defined in* (1.10) *are related by*

(2.1) $\quad {}^*g=g,\quad {}^*k=k.$

Proof. Putting ${}^*\bar{\mathcal{G}}=\mathrm{Det}({}^*g^{\lambda\nu})$, we have

(2.2) $\quad \mathcal{G}{}^*\bar{\mathcal{G}}=1,\quad {}^*\mathcal{G}={}^*\mathfrak{h}^2\,{}^*\bar{\mathcal{G}},\quad {}^*\mathfrak{h}=g^2\mathfrak{h},\quad {}^*\mathcal{K}=g^2\mathcal{K},$

which may be obtained from (1.5), (1.8)a, and (1.11)b. The relations

(2.1) are results of (1.10) and (2.2). An alternative proof of $^*k=k$ is obtained from (1.12) using $^*g=g$.

THEOREM (2.2). *We have*

$$(2.3)\qquad {}^*\begin{Bmatrix}\alpha\\ \lambda\alpha\end{Bmatrix}=\begin{Bmatrix}\alpha\\ \lambda\alpha\end{Bmatrix}+\frac{g,_\lambda}{g}\left(g,_\lambda=\frac{\partial g}{\partial x^\lambda}\right),$$

where $\begin{Bmatrix}\nu\\ \lambda\mu\end{Bmatrix}$ *and* ${}^*\begin{Bmatrix}\nu\\ \lambda\mu\end{Bmatrix}$ *are the Christoffel symbols of the second kind formed with respect to* $h_{\lambda\mu}$ *and* $^*h_{\lambda\mu}$, *respectively.*

Proof. In virtue of (1.11)b, two Christoffel symbols are related by

$$(2.4)\qquad {}^*\begin{Bmatrix}\nu\\ \lambda\mu\end{Bmatrix}=\begin{Bmatrix}\nu\\ \lambda\mu\end{Bmatrix}+\frac{1}{2g}\ (g,_\mu\delta_\lambda^\nu+g,_\lambda\delta_\mu^\nu-g,_\beta h^{\nu\beta}h_{\lambda\mu}).$$

Hence

$${}^*\begin{Bmatrix}\alpha\\ \lambda\alpha\end{Bmatrix}=\begin{Bmatrix}\alpha\\ \lambda\alpha\end{Bmatrix}+\frac{1}{2g}\ (g,_\alpha\delta_\lambda^\alpha+g,_\lambda\delta^\alpha{}_\alpha-g,_\beta\delta_\lambda{}^\beta)=\begin{Bmatrix}\alpha\\ \lambda\alpha\end{Bmatrix}+\frac{g,_\lambda}{g},$$

which proves (2.3).

THEOREM (2.3). *We have*

$$(2.5)\qquad {}^*\nabla_\nu{}^*k_{\omega\mu}=g\nabla_\nu k_{\omega\mu},$$

where ∇_ν *and* $^*\nabla_\nu$ *are the symbolic vector of the covariant derivative with respect to* $\begin{Bmatrix}\nu\\ \lambda\mu\end{Bmatrix}$ *and* ${}^*\begin{Bmatrix}\nu\\ \lambda\mu\end{Bmatrix}$, *respectively.*

Proof. Since $k_{\omega\mu}$ is skew-symmetric, it suffices to show that $^*\nabla_\nu{}^*k_{12}=g\nabla_\nu k_{12}$. This result follows in the following way, using (1.11)b and (2.3):

$$\begin{aligned}{}^*\nabla_\nu{}^*k_{12}&=\partial_\nu{}^*k_{12}-{}^*\begin{Bmatrix}\beta\\ \nu1\end{Bmatrix}{}^*k_{\beta2}-{}^*\begin{Bmatrix}\beta\\ \nu2\end{Bmatrix}{}^*k_{1\beta}\\ &=\partial_\nu{}^*k_{12}-{}^*\begin{Bmatrix}\alpha\\ \nu\alpha\end{Bmatrix}{}^*k_{12}\\ &=\partial_\nu(gk_{12})-\left(\begin{Bmatrix}\alpha\\ \nu\alpha\end{Bmatrix}+\frac{g,_\nu}{g}\right)gk_{12}=g\nabla_\nu k_{12}.\end{aligned}$$

REMARK. Chung [2] proved that in 2-dimensional g-UFT the torsion tensor $S_{\omega\mu\nu}$ satisfying Einstein's equations (1.3) is given by

$$S_{\omega\mu\nu}=\frac{1}{g}\nabla_\nu k_{\omega\mu}.$$

In virtue of (2.1) and (2.5), we see that in 2-dimensional $^*g$-UFT the same torsion tensor $S_{\omega\mu\nu}$ (satisfying (1.9)) may be given by a simple expression

$$S_{\omega\mu\nu}=\frac{1}{{}^*g^2}{}^*\nabla_\nu{}^*k_{\omega\mu}.$$

References

1. K. T. Chung & S. K. Kim, *On the algebra and Einstein's connection in 2-dimensional *g-UFT*, Yonsei Nonchong, **16**(1979)
2. K. T. Chung, *Some recurrence relations and Einstein's connection in 2-dimensional g-UFT*, Nuovo Cimento, To be published in 1980
3. V. Hlavaty, *Geometry of Einstein's unified field theory*, P. Noordhoff Ltd., 1957

Yonsei University