

ON A CLASS OF SYMMETRIC STABLE PROBABILITY MEASURES ON SEPARABLE FRÉCHET SPACES^{*)}

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1. Introduction

Let E be a real separable Fréchet space, E^* its topological dual and $\mathcal{B}(E)$ the σ -algebra generated by the class of open sets in E . By a measure on E we shall always mean a probability measure defined on $\mathcal{B}(E)$. For a measure μ on E the *characteristic functional* of μ , denoted by $\hat{\mu}(\cdot)$, is a complex valued function on E^* defined by

$$\hat{\mu}(y) = \int_E e^{i(x, y)} d\mu(x)$$

where (\cdot, \cdot) denotes the natural bilinear form on $E \times E^*$. For two measures μ, ν on E , the *convolution* of μ and ν , is defined by

$$\mu * \nu(B) = \int_E \mu(B - x) d\nu(x)$$

for every $B \in \mathcal{B}(E)$.

We say that a linear operator $S: E^* \rightarrow E$ is a *covariance operator* if it is symmetric and non-negative definite in the sense that $(Sy_1, y_2) = (Sy_2, y_1)$ and $(Sy_1, y_1) \geq 0$ for all $y_1, y_2 \in E^*$. We denote by $\mathcal{O}(E)$ the class of all those S 's corresponding to which there exists a centered gaussian measure μ on E with S its covariance. For some special cases of E such as separable Hilbert space H , R^∞ , l^p , L^p ($p \geq 1$), many authors have obtained the necessary and sufficient conditions for a covariance operator S to be in the class $\mathcal{O}(E)$; for instance, see the work of Prohorov [1] on separable Hilbert spaces, Vakhania [3], [4] on l^p and R^∞ spaces, and Rajput [2] on L^p ($p \geq 1$).

The main purpose of this paper is to identify each function of the form

$$\varphi(y) = e^{-(Sy, y)^{\alpha/2}}, \quad y \in E^*, \quad S \in \mathcal{O}(E), \quad 0 < \alpha < 2$$

with the characteristic functional of a unique symmetric stable measure on E represented by a certain mixture of Gaussian measure on E (Theorem 3.1). We then apply this result to obtain several properties on such measures; topological support (Theorem 3.2), equivalence of measures (Theorem 3.3).

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Furthermore, by specifying E to be R^∞ space, we obtain the necessary and sufficient condition for such measures on R^∞ space to be concentrated on $l^p(1 \leq p < \infty)$ by using results for Gaussian measures due to Vakhania [4].

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2. Necessary lemmas

In this section we will give the necessary lemmas which will be needed in the proof of the main theorem.

We first begin with the following definitions.

DEFINITION 2.1. A measure μ on E is called *symmetric stable of type α* , $0 < \alpha < 2$, if it is symmetric, and for any positive reals a, b , there exists $c = (a^\alpha + b^\alpha)^{1/\alpha}$ such that $T_a\mu * T_b\mu = T_c\mu$ where $T_a\mu$ is a measure on E defined by $T_a\mu(B) = \mu(1/aB)$ for every $B \in \mathcal{B}(E)$.

DEFINITION 2.2. The *topological support* of a measure μ on E , denoted by S_μ , is the smallest closed set in E with full measure.

It can be shown that S_μ is the set of all $x \in E$ for which $\mu(U) > 0$ for all open set U containing x . The first lemma is an immediate corollary of a well-known theorem of Bernstein and Widder[5].

LEMMA 2.3. Let $0 < \alpha < 2$; then there exists a probability measure P_α on $(0, \infty)$ such that for every $t \geq 0$,

$$e^{t^\alpha} = \int_0^\infty e^{-\frac{1}{2}t^2x^2} dP_\alpha(x).$$

Proof: If $0 < \beta < 1$, then e^{-t^β} is completely monotone on $[0, \infty)$ ([5], p. 854). Hence from ([5], p. 883), there exists a probability measure P_β on $(0, \infty)$ such that

$$e^{t^\beta} = \int_0^\infty e^{-tx} dP_\beta(x),$$

for all $t \geq 0$. Now let $\alpha \in (0, 2)$ and let $\beta = \alpha/2$; then

$$e^{-t^\alpha} = e^{-(t^2)^\beta} = \int_0^\infty e^{-t^2x} dP_\beta(x).$$

By an obvious change of variable formular, we obtain the desired result.

LEMMA 2.4. Let μ be a measure on E , and let P be any finite measure on $\mathcal{B}((0, \infty))$ (σ -algebra generated by open sets in $(0, \infty)$), Then for each fixed $B \in \mathcal{B}(E)$, the real function $f(x) = T_x\mu(B)$ is Borel measurable on $(0, \infty)$.

Proof: Let's consider $(0, \infty) \times E$ as the product measure space of two measure spaces $((0, \infty), \mathcal{B}((0, \infty)), P)$ and $(E, \mathcal{B}(E), \mu)$. If g denotes the scalar multiplication on E restricted to $(0, \infty) \times E$, then g is a continuous mapping, and hence a measurable mapping on $(0, \infty) \times E$. Thus for every fixed $B \in \mathcal{B}(E)$, $I_B \circ g$ is measurable from $(0, \infty) \times E$ to E , where I_B is the indicator function of B . Since

$$f(x) = \int_E I_B \circ g(x, r) d\mu(r) = T_x \mu(B),$$

it follows from Fubini's theorem that f is a measurable function on $(0, \infty)$. This completes the proof.

3. Theorems

In this section we first prove the main theorem of this paper, and by using this result, we obtain some more interesting results.

THEOREM 3.1. *Let $S \in \mathcal{O}(E)$, and let $0 < \alpha < 2$. Then the function $\varphi(y) = \exp\{-(Sy, y)^{\alpha/2}\}$ is a characteristic functional of a unique symmetric stable measure μ of type α on E ,*

Proof: For the given $\alpha \in (0, 2)$, we have, from Lemma 2.3, a probability measure P_α on $(0, \infty)$ such that

$$e^{-t^\alpha} = \int_0^\infty e^{-\frac{1}{2}t^2x^2} dP_\alpha(x), \quad t \geq 0. \tag{3.1}$$

Let ν be the centered Gaussian measure determined by the given $S \in \mathcal{O}(E)$. For every $B \in \mathcal{B}(E)$, define

$$\mu(B) = \int_0^\infty T_x \nu(B) dP_\alpha(x). \tag{3.2}$$

Then by Lemma 2.4, μ is well-defined and clearly a measure on E .

Now we shall show that φ is the characteristic functional of μ . We first note that from (3.1), for every $B \in \mathcal{B}(E)$,

$$\begin{aligned} \mu(B) &= \int_E I_B(r) d\mu(r) \\ &= \int_E \int_0^\infty I_B(r) T_x \nu(dr) dP_\alpha(x) \\ &= \int_0^\infty \int_E I_B(r) T_x \nu(dr) dP_\alpha(x). \end{aligned}$$

This shows that the equality

$$\int_E f(r) d\mu(r) = \int_0^\infty \int_E f(r) T_x \nu(dr) dP_\alpha(x) \tag{3.3}$$

holds for every indicator function f . Hence this equality holds for every

simple function. Therefore, by the easy application of monotone convergence theorem, (3.3) implies that for every $y \in E^*$

$$\begin{aligned} \int_E e^{i(r,y)} d\mu(r) &= \int_0^\infty \int_E e^{i(r,y)} T_x \nu(dr) dP_\alpha(x) \\ &= \int_0^\infty \int_E e^{i(r,xy)} \nu(dr) dP_\alpha(x) \\ &= \int_0^\infty \hat{\nu}(xy) dP_\alpha(x) \\ &= \int_0^\infty e^{-\frac{1}{2}x^2 \langle Sy, y \rangle} dP_\alpha(x) \\ &= e^{-\langle Sy, y \rangle^{\alpha/2}} \quad (\text{by (3.1)}). \end{aligned}$$

This proves that φ is the characteristic functional of the measure μ .

Since it can be easily shown that for every reals $a > 0$, $b > 0$, and $y \in E^*$, $\varphi(ay) \cdot \varphi(by) = \varphi((a^\alpha + b^\alpha)^{1/\alpha} y)$, it follows that μ is a symmetric stable measure on E of type α . Moreover, since it is known [1] that characteristic functionals determines uniquely measures on E , we complete the proof.

In view of Theorem 3.1, if a measure μ on E has the characteristic functional of the form $\hat{\mu}(y) = e^{-\langle Sy, y \rangle^{\alpha/2}}$, $y \in E^*$, where $S \in \mathcal{O}(E)$, $\alpha \in (0, 2)$, we shall say that it is the symmetric stable measure of type α determined by $S \in \mathcal{O}(E)$.

THEOREM 3.2. *Let μ_α and ν be the symmetric stable measure of type $\alpha \in (0, 2)$ determined by a fixed $S \in \mathcal{O}(E)$ and the centered Gaussian measure corresponding to S , respectively. Then $S_{\mu_\alpha} = S_\nu$ for all $\alpha \in (0, 2)$.*

Proof: Let $\alpha \in (0, 2)$. Since S_ν is a closed subspace of E ([2]), we have $T_x \nu(S_\nu) = \nu(S_\nu) = 1$; whence from (3.2), $\mu_\alpha(S_\nu) = 1$. Moreover, if $r \in S_\nu$ and V is an open neighborhood of r , then $\frac{1}{x}r \in S_\nu$ and $\frac{1}{x}V$ is an open neighborhood of $\frac{1}{x}r$ for every $x > 0$. It follows that $\nu\left(\frac{1}{x}V\right) = T_x \nu(V) > 0$ for every $x > 0$. Therefore, from (3.2), we have $\mu_\alpha(V) > 0$. Thus S_ν is the topological support of μ_α . This completes the proof.

THEOREM 3.3. *Let μ_1 and μ_2 be two symmetric stable measures of type $\alpha \in (0, 2)$ determined by $S_1, S_2 \in \mathcal{O}(E)$ respectively, and let ν_1 and ν_2 be the centered Gaussian measures corresponding to $S_1, S_2 \in \mathcal{O}(E)$. Then $\nu_1 \sim \nu_2$ implies $\mu_1 \sim \mu_2$, where \sim denotes equivalence of measures.*

Proof: We first note that $\nu_1 \sim \nu_2$ implies $T_x \nu_1 \sim T_x \nu_2$ for all $x > 0$. Let $B \in \mathcal{B}(E)$ such that $\mu_1(B) = 0$; then from (3.2), $\mu_1(B) = 0$ implies $T_x \nu_1(B) = 0$ a. e. $[P_\alpha]$. Since $T_x \nu_1 \sim T_x \nu_2$ for all $x > 0$ it follows that $T_x \nu_2(B) = 0$ a. e.

$[P_\alpha]$. Hence we have $\mu_2(B) = 0$. This shows that $\mu_2 \ll \mu_1$. By the similar arguments, we can show that $\mu_1 \ll \mu_2$. Hence $\mu_1 \sim \mu_2$. This completes the proof.

Now let $E = R^\infty$ be the separable Fréchet space of all real sequences with the metric defined by

$$d(x, y) = \sum_{k=1}^{\infty} \alpha_k \frac{|x_k - y_k|}{1 + |x_k - y_k|}$$

where α_k are positive real numbers which forms a convergent series. A characterization of the centered Gaussian measure on l^p space, $1 \leq p < \infty$, in terms of its covariance operator S has been known in [4]. By using this result, we extend it to the case of symmetric stable measures determined by $S = (s_{ij}) \in \mathcal{O}(E)$. Let c_0 and l_∞ denote respectively,

$$c_0 = \{x : \lim_{j \rightarrow \infty} x_j = 0\}, \text{ and } l_\infty = \{x : \sup_j |x_j| < \infty\}.$$

It can be shown that $c_0, l_\infty \in \mathcal{B}(R^\infty)$.

THEOREM 3.4. *Let μ_α be the symmetric stable measure on R^∞ of type $\alpha \in (0, 2)$ determined by $(s_{ij}) \in \mathcal{O}(R^\infty)$. Then*

- (a) $\mu_\alpha(l^p) = 1$, $1 \leq p < \infty$, if and only if $\sum_{i=1}^{\infty} s_{ii}^{\frac{p}{2}} < \infty$.
- (b) if $\sum_{i=1}^{\infty} e^{-\frac{\varepsilon}{s_{ii}}} < \infty$ for all $\varepsilon > 0$, then $\mu_\alpha(c_0) = 1$.
- (c) if there exists an $\varepsilon > 0$ such that $\sum_{i=1}^{\infty} e^{-\frac{\varepsilon}{s_{ii}}} < \infty$, then $\mu_\alpha(l_\infty) = 1$.

Proof: Let ν be the centered Gaussian measure corresponding to $S = (s_{ij})$. Since it is known in [4] that $\nu(l^p) = 1$ if and only if $\sum_{i=1}^{\infty} s_{ii}^{\frac{p}{2}} < \infty$, it follows from (3.2) that $\mu_\alpha(l^p) = 1$ if and only if $\sum_{i=1}^{\infty} s_{ii}^{\frac{p}{2}} < \infty$. The proofs of (b) and (c) immediately follows from the known results in [3] that if $\sum_{i=1}^{\infty} e^{-\varepsilon/s_{ii}} < \infty$, for all $\varepsilon > 0$, then $\nu(c_0) = 1$, and that if there exists $\varepsilon > 0$ such that $\sum_{i=1}^{\infty} e^{-\varepsilon/s_{ii}} < \infty$, then $\nu(l_\infty) = 1$.

REMARK 3.5. We note that no special property of the probability measure P_α has been used in the proofs of Theorems 3.2, 3.3 and 3.4; thus these results will hold for any mixture of Gaussian measures on E .

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