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THE KOBAYASHI AND CARATHEODORY PSEUDOMETRICS ON SURFACES

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1. Introduction

The Kobayashi and Caratheodory pseudometrics are two extremal metrics satisfying the Schwarz Pick System. In this paper we investigate the differences of two pseudometrics on surfaces, and we can show that they are different on hyperbolic surfaces. The proof highly depends on the invariant form of Schwarz lemma. It was impossible to obtain similar results in higher dimmensional case since the Schwarz lemma is not so powerful in the higher dimmension.

We list here several definitions which will be used in our discussion. Let P be the Poincaré metric defined on the unit open disk D in the complex plane and let S be a complex manifold then we consider the real valued function $C: S\times S\rightarrow R$;

$$C(x, y)=\sup\{P(g(x), g(y))\,|\,g\in G\}$$

where G is the set of holomorphic functions $g: S\rightarrow D$. The function C is called the *Caratheodory Pseudometric* on S.

We define the Kobayashi pseudometric K on a complex manifold as follows. Given two points x and y in S, we choose points

$$x=x_0,\quad x_1, \dots, x_{k-1}, x_k=y$$

of S, points $a_1, a_2, \dots, a_k$, $b_1, b_2, \dots, b_k$ of D, and f_k holomorphic mappings $f_1, f_2, \dots, f_k$ of D into S such that $f_i(a_i)=x_{i-1}$ and $f_i(b_i)=x_i$ for $i=1, 2, \dots, k$. For each choice of points and mappings thus made, we consider the number

$$P(a_1, b_1)+\cdots+P(a_k, b_k).$$

Let $K(x, y)$ be the infimum of the numbers obtained in this manner for all possible choices. The function K is called *Kobayashi Pseudometric* for S. For a complex manifold S and K the pseudometric defined above, if K is a metric, that is $K(x, y)>0$ for all $x\neq y$, then S is called a ***hyperbolic manifold***. It is well known that $K(x, y)\geq C(x, y)$ for any x and y in S.

For the definitions of K and C and other relevant results about pseudometrics, one can refer [3], [4] or [1]. And also we use the following properties of the pseudometrics. If f is a holomorphic mapping from a complex manifold S to another complex manifold T, then we know that for any x, y in S and $f(x)$, $f(y)$ in T, K and C satisfies $K(x,y)\geq K(f(x),f(y))$ and $C(x,y)\geq C(f(x),f(y))$.

2. Main results

It is clear that Poincaré metric, Caratheodory pseudometric and Kobayashi pseudometrics are identical on the unit open disk D in the complex plane. We will use the following form of the Scwarz lemma in our proof.

LEMMA. *Let K be the Kobayashi pseudometric on the unit open disk on the complex plane. Let $f: D\rightarrow D$ be a holomorphic map then the pseudometric K decreases by f that is;*

$$K(x,y)\geqq K(f(x),f(y)),$$

and the equality at a singlepair of $x\neq y$ of D implies that f is an automorphism of D.

For the lemma one can consult [2].

We need the following generalization of the Schwarz lemma.

LEMMA. *Let f be a holomorphic map of the unit disk D of the complex plane into a Riemann surface S. If there exists two different points x and y in D with*

$$K(x,y)=K(f(x),f(y)),$$

then f is a covering map.

Proof. First we assume that the space S admits D as the holomorphic covering surface. Let $h: D\rightarrow S$ be a holomorphic covering map. Let k be the holomorphic function maps D into D and satisfies $h\circ k=f$. Then we have;

$$K(x,y)\geqq K(h(k(x)),\ h(k(y)))=K(f(x),f(y).$$

The Pseudometric decreasing property of holomorphic functions and the hypothesis of the lemma give the following equalities;

$$K(x,y)=K(k(x),k(y))=K(h(k(x)),\ h(k(y))).$$

By the above Schwarz lemma we know that k is an automorphism of D. Hence the function $f(\doteqdot h\circ k)$ is also a covering map. Now consider the other remaining case; the complex plane is a holomorphic covering surface of S. Then since K is identically zero pseudometric for the complex plane, we know that K must be identically zero for S. Hence there is no holomorphic function satisfying the hypothesis of the lemma, and the proof is complete.

Let S be a complex manifold which has the unit disk D as a covering surface with holomorphic covering map $f: D \to S$, then it is known that for any x and y in S,

$$K(x, y) = \inf\{K(a, b) \mid f(a) = x, f(b) = y \text{ and } a, b \in D\}$$

Hence the above lemma is the converse of this assertion. For a proof of this results, see [3] and [4], there they prove that the same is true for a complex manifold with hyperbolic covering space.

By the above lemma and the comments following it, we know that for a hyperbolic surface the Kobayashi metric is always given by a covering map, we write it as a corollary.

COROLLARY. *Let S be a hyperbolic surface then for any x and y in S there is a covering holomorphic map f of the open unit disk D in the complex plane into S satisfying*

$$K(x, y) = K(a, b), f(a) = x \text{ and } f(b) = y.$$

And conversely if a map f satisfies the above equality for some $x \neq y$ then it is a covering map.

Now we prove the main theorem which gives the comparision of the Kobayashi and the Caratheodory pseudometric.

THEOREM. *Let S be a hyperborlic surface and let K and C be the Kobayashi and the Caratheodory pseudometric respectively then $K(x, y) \geqq C(x, y)$ for any pair $x \neq y$. The equality holds for any pair of elements if and only if S is holomorphically equivalent to the open unit disk in the complex plane.*

Proof. The inequality is true in any case, hence assume the equality exists for a pair. Let A be the family of all holomorphic functions f from S into the open unit disk D which satisfy $f(x) = 0$ and $f(y) \geq 0$. Then by the definition of Caratheodory pseudometric we have the following

$$C(x, y) = \sup\{K(0, a) \mid g(x) = 0, g(y) = a, g \in A\}.$$

Here K denotes the Kobayashi pseudometric. We can choose a sequence of holomorphic functions in A such that

$$C(x, y) = \lim_{n \to \infty} K(g_n(x), g_n(y)).$$

Since D is bounded we can choose a subsequence of $\{g_n\}$ which converges uniformly on compact subsets of S into D and we have the following equalities;

$$K(x, y) = K(g(x),\ g(y)) = C(x, y).$$

Let $h: D \to S$ be a covering holomorphic map of D into S. Then we can find a and b in D with

$$K(a, b) = K(h(a), h(b)),\ h(a) = x \text{ and } h(b) = y.$$

Consider the composition map $g \circ h: D \to D$. Then we have

$$K(a, b) = K(h \circ g(a),\ h \circ g(b)).$$

By the above lemma we know that $h \circ g$ is a covering map, hence it must be an automorphism of D. Since $h \circ g$ is an automorphism, we have that $h \circ g$ is an one to one map and h must be one to one. We infer that S is holomorphically equivalent to D. This proves our theorem.

By the theorem we know that the Kobayashi and Caratheodory pseudometrics are different on any hyperbolic surface which is not simply connected. We list this results as a corollary.

COROLLARY. *Let S be a non-simply connected hyperbolic surface. Then $K > C$; that is*

$$K(x, y) > C(x, y)$$

for any two different points x and y in S.

Now we consider the fixed point properties of holomorphic mapping as an application of the above discussions.

LEMMA. *Let D be the open unit disk in the complex plane and let f be a holomorphic map of D into itself. If f has two fixed points then it is the identity map.*

Proof. Let x and y be the two different points which are fixed under the map f, then we have $K(x, y) = K(f(x), f(y))$.
Hence by the Schwarz lemma we know that f is an automorphism of the open disk D. Hence we know that f is an isometric map of D under the Kobayashi metric. Since the Riemanian manifold has the shortest geodesic r connecting x and y, hence f must map the geodesic r into itself. Now for any two points s and t on r we know that

$$K(s, t) = K(f(s),\ f(t)) \text{ and } f\{(s), f(t)\} \subset r.$$

Hence f must fix all points on r and since f is holomorphic it must be the identity map.

In the above proof we may avoid the use of differential geometric idea, simply appealing to the known equations of an automorphism on D, but we prefer this proof since the same idea can be applied to certain Riemann surfaces. Also one can see that the above lemma is another form of the Schwarz lemma.

DEFINITION. A Riemann surface S is called *rigid* if for any x and y on S there exists the shortest (unique) geodesic connecting these points.

We can say some what similar results about the rigid Riemann surfaces as the above lemma.

THEOREM. *Let S be a hyperbolic Riemann surface and let f be a holomorphic self map of S. If there exist two different points x any y on S such that*

$$K(x, y)=K(f(x), f(y)).$$

Then f is an automorphism. If further f fixes x and y and S is rigid space, then f is the identity map.

Proof. Let x and y be those given two points on S. Let $h: D \to S$ be a holomorphic covering map of the unit disk D onto S and let g be the lifting of f such that $f=h\circ g$.

We choose two points s, t in D with $h(s)=x$, $h(t)=y$ and $K(s, t)=K(x, y)$. Then we have

$$K(s, t)=K(x, y)=K(f(s), f(t))$$

and

$$K(s, t)=K(g(s), g(t))=K(h\circ g(s),\ h\circ g(t)).$$

Hence by the Schwarz lemma we know that g is an automorphism of D. Since g is an automorphism, it follows that f is also an automorphism of S.

Now suppose that $y(x)=x$, $f(y)=y$ and S is rigid respect the metric K. Since f is an automorphism, it is an isometry of S, and hence f must fix the shortest geodesic connecting x and y. Moreover, the map is holomorphic and it must be the identity map of S. Thus we proved the theorem.

REMARK. The first assertion of the above theorem may be considered as a Schwarz lemma on Riemann surfaces.

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