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ON THE WREATH PRODUCTS

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1. Introduction

In this paper, we study some properties of the wreath product of groups, and using these properties we give a new proof of the Schur-Zassenhaus Theorem. In section 2, we will discuss some important properties of the wreath product and prove the following universal mapping property.

THEOREM 2.3. *Let K and H be groups. Let $\alpha : K \to K_r^{\wr}H$ be a mapping defined by $\alpha(k)=(e_H, f_k)$, where $f_k(e_H)=k$ and $f_k(h)=e_K$ for all $h\neq e_H$. And let $\beta : H \to K_r^{\wr}H$ be a mapping defined by $\beta(h)=(h, e_F)$. Then α and β are injective homomorphisms.*

Moreover, given any group G and homomorphisms $\varphi : K\to G$ and $\psi : H\to G$ such that

$$[\varphi(K)^{\psi(h)}, \varphi(K)^{\psi(h')}]=1$$

for all $h, h'\in H$, there exists a unique homomorphism $\eta : K_r^{\wr}H \to G$ such that

$$\eta\cdot\alpha=\varphi \text{ and } \eta\cdot\beta=\psi.$$

The existence of η is suggested by [3], when H and K are finite. In this theorem, the universal mapping property of η is established in the general case.

In section 3, we will prove Theorem 3.2 (Schur-Zassenhaus Theorem) by using the properties of the wreath product. There are several methods in proving this theorem. Originally Schur proved only (2) of Theorem 3.2 and later H. Zassenhaus proved (1) of Theorem 3.2 using an inductive argument based on Schur's result. This theorem can be also derived by calculating the cohomology groups Infact, $H^1(G, N)=1$ implies (1) and $H^2(G, N)=1$ implies (2). Here we shall give another proof of (2). The key of the proof is to use the conjugacy property (1) and the properties of the wreath product.

The notation of this paper is standard. It is taken from [3] and [5]. We denote the center of G by $Z(G)$ and the centralizer of a subgroup H in G by $C_G(H)$. A subgroup H of a finite group G is called a Hall subgroup if $|H|$ and $|G : H|$ are relatively prime. A subgroup H is called a complement of a subgroup N in G if $G=NH$ and $N\cap H=1$.

2. Some properties of the wreath products

Let H be a permutation group on a set X. For any group K, the wreath product of K and H is defined as follows:

Let F be the set of all functions defined on X taking values in K. Then F is a group under the multiplication given by

$$(fg)(i)=f(i)g(i),\quad i\in X.$$

An action of H on F is induced by the formula

$$f^h(i)=f(h^{-1}(i)),\quad i\in X.$$

It is easy to check that H is a group of automorphisms of F. The semi-direct product of F and H with respect to the action defined as above is *the unrestricted wreath product* of K and H, which is denoted by $K\mathrm{Wr}H$. *The restricted wreath product*, denoted by $K\mathrm{wr}H$, is the semi-direct product of F_0 and H where F_0 is the subgroup of F consisting those functions f which satisfy $f(i)=e_K$ for all but finitely many elements of X.

Since F_0 is an H-invariant subgroup of F, the restricted wreath product is a subgroup of the unrestricted wreath product. And it is clear that $K\mathrm{wr}H=K\mathrm{wr}H$ if and only if X is finite or K is trivial.

If H is not presented as a permutation group, then we consider H as a permutation group on the set $X=H$ induced by left multiplication. Thus if $h\in H$, h acts on $x\in X$ as $h(x)=h^{-1}x$. In this case, the unrestricted [resp., restricted] wreath product of K and H is called *the standard unrestricted* [resp., *standard restricted*] *wreath product*, and denoted by $K\wr H$ [resp., $K\wr_r H$]. Thus $K\wr H$ [resp., $K\wr_r H$] is the set of all pairs (h,f) where $h\in H$ and f is a function defined on H taking values in K [resp., satisfying $f(x)=e_K$ for all but finitely many elements x of X].

We have

$$(h,f)(u,g)=(hu,f^u g),$$

where $f^u g(x)=f(u^{-1}(x))g(x)=f(ux)g(x)$ for $x\in X$. And if K and H are finite groups, then it is clear that $K\wr H=K\wr_r H$ and $|K\wr H|=|K|^{|H|}|H|$.

The following lemma which illustrates the structures of $K\wr H$ and $K\wr_r H$ will be useful in the next section.

LEMMA 2.1. *The standard unrestricted* [resp., *restricted*] *wreath product of K and H contains a normal subgroup F^* and a subgroup H^* satisfying the properties:*

(1) $G=F^*H^*$, $F^*\cap H^*=(e_H,e_F)$ *and* $G/F^*\cong H^*\cong H$,

(2) F^* *is isomorphic to the direct product* [resp., *direct sum*] *of* $|H|$ *copies of* K.

Proof. First, assume that G is the standard unrestricted wreath product of K and H. Let $H^*=\{(h, e_F)\,|\,h\in H\}$, where $e_F(x_i)=e_K$ for all $x_i\in H$. Then clearly $H^*\cong H$. Let

$$F^*=\{(e_H, f)\,|\,f : H\to K\},$$

and

$$F_{x_i}=\{(e_H, f)\,|\,f(x_j)=e_K \text{ for } x_j\neq x_i\}.$$

Then $F_{x_i}\cong K$ and $F^*=\mathop{\Pi}_{x\in H} F_{x_i}$.

The mapping $\varepsilon : G\to H$ defined by $\varepsilon(h, f)=h$ *is* an epimorphism of G onto H with kernel F^*. Hence $G/F^*\cong H^*\cong H$. Since

$$(h, f)=(e_H, f^{h^{-1}})\,(h, e_F),$$

we have $F^*H^*=G$ and $F^*\cap H^*=\{(e_H, e_F)\}$.

We can give a similar proof for the standard restricted wreath product.

LEMMA 2. 2. *Let G be the (standard) unrestricted wreath product of group K and H. For a subgroup L of K, let $D(L)$ be the subgroup of G such that*

$$D(L)=\{(e_H, f)\,|\,f : H\to K \text{ is a constant function taking values in } L\}.$$

Then

(1) $C_G(H)=Z(H)D(K)$, *and*

(2) $Z(G)=Z(D(K))$.

In particular, if G is the restricted wreath product of K and H, and $|H|$ is infinite, then $Z(G)$ is trivial.

Proof. (1) Let $(h, f)\in C_G(H)$. Then

$$(h, f)\,(h', e_F)=(h', e_F)=(h', e_F)\,(h, f)$$

for all $h'\in H$. Thus we have $hh'=h'h$ and $f(h'h)=f(h)$ for all $h'\in H$. Hence $h\in Z(H)$ and f is constant on H. This implies $C_G(H)=Z(H)D(K)$. The converse argument is also valid.

(2) If $(h, f)\in Z(G)$, then $(h, f)\,(h', f')=(h', f')\,(h, f)$ for all $(h', f')\in G$. Hence $f^{h'}f'=f'^{h}f$. In particular, $ff'=f'^{h}f$ for all f', and $f^{h'}=f$ for all h', that is, f is a constant function. Suppose that $h\neq e_H$. Let $f(e_H)=k\in K$. Then there exists a function $f' : H\to K$ such that $f'(e_H)=k^{-1}$ and $f'(h)\neq k^{-1}$. For this function f', $(ff')\,(e_H)=e_K$ but $(f'^{h}f)\,(e_H)=f'(h)f(e_H)\neq k^{-1}k=e_K$. Therefore $ff'\neq f'^{h}f$. This contradicts to $ff'=f'^{h}f$. Hence $h=e_H$, which implies $(h, f)\in D(K)$ and $ff'=f'f$ for all f'. Therefore, we have

$$Z(G)=D(K)\cap C_G(F^*)=Z(D(K)),$$

where $F^*=\{(e_H, f')\,|\,f' : H\to K \text{ function}\}$.

If H is infinite group and G is restricted wreath product of K and H, then $D(K)=\{(e_H, e_F)\}$. Therefore, $Z(G)=Z(D(K))$ is trivial.

THEOREM 2.3. *Let K and H be groups. Let $\alpha : K \to K_r^{\wr} H$ be a mapping defined by $\alpha(k)=(e_H, f_k)$, where $f_k(e_H)=k$ and $f_k(h)=e_K$ for all $h \neq e_H$. And let $\beta : H \to K_r^{\wr} H$ be a mapping defined by $\beta(h)=(h, e_F)$. Then α and β are injective homomorphisms.*

Moreover, given any group G and homomorphisms $\varphi : K \to G$ and $\phi : H \to G$ such that

$$[\varphi(K)^{\phi(h)},\ \varphi(K)^{\phi(h')}]=1$$

for all $h,\ h' \in H$, there exists a unique homomorphism $\eta : K_r^{\wr} H \to G$ such that

$$\eta \cdot \alpha=\varphi \text{ and } \eta \cdot \beta=\phi.$$

Proof. It is clear that α and β are injective homomorphisms.

We define a homomorphism η from $K_r^{\wr} H$ into G by

$$\eta((h,f))=\phi(h) \prod_{x \in H} \varphi(f(x))\phi(x^{-1}).$$

Then η is well defined and

$$\eta(h,f)(h',f'))=\eta((hh',\ f^{h}f'))=\phi(hh') \prod_{x \in H} \varphi(f^{h}f'(x))^{\phi(x^{-1})}$$

By the definition of the restricted wreath product, there exist only finitely many x's, say $x_1, \ldots, x_n$, in H such that $f^{h}f'(x_i) \neq e_K$. Hence

$$\begin{aligned}\eta(h,f)(h',f'))&=\phi(h)\phi(h')\varphi(f^{h}f'(x_1))^{\phi(x_1^{-1})}\cdots\varphi(f^{h}f'(x_n))^{\phi(x_n^{-1})}\\&=\phi(h)\phi(h')(\varphi(f(h'x_1)))^{\phi(x_1^{-1})}\cdots(\varphi(f(h'x_n)))^{\phi(x_n^{-1})}\\&\qquad(\varphi(f'(x_1)))^{\phi(x_1^{-1})}\cdots(\varphi(f'(x_n)))^{\phi(x_n^{-1})}\\&=\phi(h)(\varphi(f(h'x_1)))^{\phi((h'x_1)^{-1})}\cdots(\varphi(f(h'x_n)))^{\phi((h'x_n)^{-1})}\\&\qquad\phi(h')(\varphi(f'(x_1)))^{\phi(x_1^{-1})}\cdots(\varphi(f'(x_n)))^{\phi(x_n^{-1})}\\&=\eta((h,f))\eta((h',f)).\end{aligned}$$

Thus η is a homomorphism.

Next, we will show that $\eta \cdot \alpha=\varphi$ and $\eta \cdot \beta=\phi$. In fact, we have

$$\begin{aligned}(\eta \cdot \alpha)(k)&=\eta(\alpha(k))=\eta((e_H,f_k))\\&=\phi(e_H) \prod_{x \in H} \varphi(f_k(x))^{\phi(x^{-1})}=\varphi(k).\end{aligned}$$

Similarly, we have $(\eta \cdot \beta)(h)=\eta(\beta(h))=\eta((h, e_F))=\phi(h)$.

Finally, we assert the uniqueness of η. Let $\eta' : K_r^{\wr} H \to G$ be another homomorphism with $\eta' \cdot \alpha=\varphi$ and $\eta' \cdot \beta=\phi$. Then for each k in K

$$\eta((e_H,f_k))=\eta(\alpha(k))=\varphi(k)=\eta'(\alpha(k))=\eta'((e_H,f_k)),$$

and for each h in H

$$\eta((h,e_F))=\eta'((h,e_F)).$$

Also, for a given $(h,f) \in K_r^{\wr} H$, let x_i be the finitely many elements of H such that $f(x_i)=k_i,\ k_i \neq e_K$ and let $f_i : H \to K$ be defined by $f_i(x_i)=k_i$ and $f_i(x)=e_K$ for all $x \in H,\ x \neq x_i$. Then $f=f_1 \cdots f_n$ and

$$(h,f)=(h,e_F)(e_H,f)=(h,e_F)(e_H,f_1)\cdots(e_H,f_n).$$

Since $(e_H,f_i)=(x_i,f_{k_i})(x_i^{-1},e_F)=(x_i,e_F)(e_H,f_{k_i})(x_i^{-1},e_F)$ for each $i=1,\ldots,n$, we have $\eta((h,f))=\eta'((h,f))$. Hence $\eta'=\eta$.

Thus the theorem is proved.

3. Schur-Zassenhaus Theorem

In this section, we will give a new proof of Schur-Zassenhaus theorem. To do this, we need the following well-known theorem(*cf.* [5]).

THEOREM 3.1. (Kaloujnine and Krasner). *Let G be a group with a normal subgroup N. Set $H=G/N$. Then G is isomorphic to a subgroup of the standard unrestricted wreath product of N and H.*

THEOREM 3.2. (Schur-Zassenhaus). *Let G be a finite group, and let N be a normal abelian Hall subgroup of G. Then*

(1) *If K is a complement of N in G, and L is a subgroup of G such that $G=NL$, then $c^{-1}Kc\subseteq L$ for some $c\in N$. In particular, if L is also a complement of N in G, then L is conjugate to K in G.*

(2) *There exists a subgroup K of G such that $G=NK$ and $N\cap K=1$.*

Proof. The proof of (1) is given in [1], but we include it here for the completeness. Here, we put

$$|N|=m \quad \text{and} \quad |G:N|=|K|=n.$$

First, consider the case $N\cap L=1$. Then K and L are complete sets of coset representatives for N in G, and so there is an isomorphism φ on K onto L such that $xN=\varphi(x)N$ for all $x\in K$. Because N is an abelian group, and $x^{-1}\varphi(x)\in N$ for all $x\in K$, the product $b=\prod_{x\in K} x^{-1}\varphi(x)$ is a well-defined element of N. Since N is a normal subgroup, therefore, for each $y\in K$,

$$(*)\qquad y^{-1}by=\prod_{x\in K}\{(xy)^{-1}\varphi(xy)\varphi(y)^{-1}y\}$$
$$=\{\prod_{x\in K}(xy)^{-1}\varphi(xy)\}\{\varphi(y)^{-1}y\}^n=b\{\varphi(y)^{-1}y\}^n,$$

because xy runs through K as x runs through K.

Since N is a Hall subgroup, m is relatively prime to n and so $ms+nt=1$ for some integers s and t. Thus for each $a\in N$, $a^{nt}=a^{1-ms}=a\cdot a^{-ms}=a$. And from (*) we get

$$y^{-1}b^ty=(y^{-1}by)^t=b^t\{\varphi(y)^{-1}y\}^{nt}=b^t\varphi(y)^{-1}y.$$

Hence $b^{-t}yb^t=\varphi(y)$ for all $y\in K$, and so $c^{-1}Kc=L$ with $c=b^t\in N$. This proves (1) for this special case.

In the general case, $N_1=N\cap L$ is normal in L (because N is normal in G), and N_1 is normal in N (because N is abelian), and so N_1 is normal in $NL=G$. The normal abelian Hall subgroup N/N_1 of G/N_1 has two complements KN_1/N_1 and L/N_1 in G/N_1. By the result proved above, $c^{-1}KN_1c/N_1=L/N_1$ for some $c\in N$, and so $c^{-1}Kc\subseteq c^{-1}KN_1c=L$. Thus the assertion (1) holds.

We now prove (2) by using the conjugacy property (1). By Theorem 3.1, G is isomorphic to a subgroup of $N\backslash G/N=G_1$ under the isomorphism θ, which was defined in the proof of Theorem 3.1. By Lemma 2.1, G_1 contains a normal subgroup

$$F^*=\{(N,f)\,|\,f : G/N\to N\}$$

and a subgroup

$$H^*=\{(g_N,\ e_F)\,|\,g_N\in G/N\}\cong G/N$$

such that $G_1=F^*H^*$ and $F^*\cap H^*=\{(N,e_F)\}$.

We then note that the following hold:

(i) $F^*\cap\theta(G)=\theta(N)$.

(ii) F^* is a normal abelian subgroup of G_1.

(iii) $|G_1 : F^*|=|G : N|=|\theta(G) : \theta(N)|=|\theta(G) : \theta(G)\cap F^*|$, and $|F^*|=|N|^n$.

Thus F^* is a Hall subgroup of G_1.

(iv) H^* is a complement of F^* in G_1.

(v) $F^*\theta(G)=G_1$.

Because of (ii), (iii) and (iv), we may apply (1) to the group G_1 to conclude that $c^{-1}H^*c\subseteq\theta(G)$ for some $c\in F^*$. Then $c^{-1}H^*c$ is clearly a complement of $\theta(N)$ in $\theta(G)$. Since θ is an isomorphism of G onto $\theta(G)$, it follows that N has a complement in G.

Thus we have completed the proof of Theorem 3.2.

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