Bull. Korean Math. Soc.
Vol. 17, No. 1, 1980

ON THE SEQUENCE SPACES $Ces(p_n)_\infty$, $0<p\leqslant 1$

BY KWANG HO SHON

§1. Introduction

Let S denotes the linear space of all infinite sequences $x=\{x_n\}$ of complex numbers over complex field. For a sequence $\{p_n\}$ of real numbers with $0<p_n\leqslant 1$ for all n, we define the sequence space

$$Ces(p_n)_\infty=\{\{x_n\} : \{x_n\}\in S,\ \sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{p_n}\right\}<\infty\}$$

with a metric defined by

$$\sigma(p_n)(x,y)=\sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^{n}|x_k-y_k|\right)^{p_n}\right\}.$$

We write $v(p_n)$ for the topology on $Ces(p_n)_\infty$ induced by the metric $\sigma(p_n)$. If $p_n=1$ for all n, we write $Ces(\infty)$ for $Ces(1)_\infty$, $v(\infty)$ for $v(1)$.

In this paper we investigate the extent to which author's paper [4] for Cesàro sequence spaces $(Ces(p), u(p))$, $1<p<\infty$, have analogues for the sequence spaces $(Ces(p_n)_\infty,\ v(p_n))$, $0<p_n\leqslant 1$.

§2. Properties of $Ces(p_n)_\infty$, $0<p_n\leqslant 1$

In this section we investigate some relationships between the $Ces(p_n)_\infty$ spaces.

LEMMA 2.1. *$(Ces(p_n)_\infty,\ v(p_n))$ is a complete metric additive topological group, and for a fixed scalar λ, the function $x\longrightarrow\lambda x$ is continuous. Moreover, the function $(\lambda,x)\longrightarrow\lambda x$ is continuous at $(\lambda,x)=(0,0)$.*

Proof. We first show that $(Ces(p_n)_\infty,\ v(p_n))$ is a complete metric additive topological group; if $\{x_n\}$, $\{y_n\}\in Ces(p_n)_\infty$, then

$$\left(\frac{1}{n}\sum_{k=1}^{n}|x_k+y_k|\right)^{p_n}\leqslant\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|+\frac{1}{n}\sum_{k=1}^{n}|y_k|\right)^{p_n}$$

$$\leqslant\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{p_n}+\left(\frac{1}{n}\sum_{k=1}^{n}|y_k|\right)^{p_n}.$$

Hence it follows that

$$(2.1)\quad \sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^{n}|x_k+y_k|\right)^{p_n}\right\}$$
$$\leqslant \sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{p_n}\right\}+\sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^{n}|y_k|\right)^{p_n}\right\}<\infty.$$

This shows that $\{x_n\}+\{y_n\}\in Ces(p_n)_\infty$. Hence $(Ces(p_n)_\infty, +)$ is a group.

It is immediate from the definition of $\sigma(p_n)$ and (2.1) that $\sigma(p_n)$ is a metric on $Ces(p_n)_\infty$. The continuity of the addition follows from the inequality; for $x, y, s, t\in Ces(p_n)_\infty$

$$\sigma(p_n)(x+y, s+t)\leqslant\sigma(p_n)(x, s)+\sigma(p_n)(y, t).$$

We have shown that $(Ces(p_n)_\infty, v(p_n))$ is a metric additive topological group.

Now we will show that $(Ces(p_n)_\infty, v(p_n))$ is complete. Let $\{x^{(n)}\}_{n=1}^{\infty}$ be a Cauchy sequence in $Ces(p_n)_\infty$ with respect to $\sigma(p_n)$. We first show that for each k, $\{x_k^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in the set of all complex numbers **C**. To show this, let k be fixed and $\varepsilon>0$. Choose a positive integer N_0 such that $k\leqslant N_0$. Since $\{x^{(n)}\}_{n=1}^{\infty}$ is Cauchy in $Ces(p_n)_\infty$, there exists n_0 such that if $n, m\geqslant n_0$, then

$$\sigma(p_N)(x^{(n)}, x^{(m)})<\left(\frac{\varepsilon}{N_0}\right)^{p_{N_0}}$$
$$\Longrightarrow \sup_N\left\{\left(\frac{1}{N}\sum_{k=1}^{N}|x_k^{(n)}-x_k^{(m)}|\right)^{p_N}\right\}<\left(\frac{\varepsilon}{N_0}\right)^{p_{N_0}}$$
$$\Longrightarrow \left(\frac{1}{N_0}\sum_{k=1}^{N_0}|x_k^{(n)}-x_k^{(m)}|\right)^{p_{N_0}}<\left(\frac{\varepsilon}{N_0}\right)^{p_{N_0}}$$
$$\Longrightarrow \sum_{k=1}^{N_0}|x_k^{(n)}-x_k^{(n)}|<\varepsilon$$
$$\Longrightarrow |x_k^{(n)}-x_k^{(m)}|<\varepsilon \text{ for each } 1\leqslant k\leqslant N_0.$$

Since k is arbitrary, it follows that for each k, $\{x_k^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in **C**. Hence for each k, $\lim_{n\to\infty} x_k^{(n)}$ exists. Thus if $\lim_{n\to\infty} x_k^{(n)}=x_k$, then we obtain a sequence $\{x_k\}_{k=1}^{\infty}$.

Now we show that $\{x_k\}\in Ces(p_n)_\infty$. Since $\{x_k^{(n)}\}_{n=1}^{\infty}$ is Cauchy, for given $\varepsilon>0$, there exists a positive integer n_0 such that for $n, m\geqslant n_0$, we have

$$\frac{1}{N}\sum_{k=1}^{N}|x_k^{(n)}-x_k^{(m)}|<\varepsilon \text{ for each } N.$$

Letting $m\longrightarrow\infty$, we obtain

$$\frac{1}{N}\sum_{k=1}^{N}|x_k^{(n)}-x_k|\leqslant\varepsilon \text{ for each } N\ (n\geqslant n_0).$$

Hence we have

$$\left(\frac{1}{N}\sum_{k=1}^{N}|x_k|\right)^{p_N}$$
$$=\left(\frac{1}{N}\sum_{k=1}^{N}|x_k-x_k^{(n)}+x_k^{(n)}|\right)^{p_N}$$
$$\leqslant\left(\frac{1}{N}\sum_{k=1}^{N}|x_k-x_k^{(n)}|\right)^{p_N}+\left(\frac{1}{N}\sum_{k=1}^{N}|x_k^{(n)}|\right)^{p_N}$$

for each N, and hence

$$\sup_N\left\{\left(\frac{1}{N}\sum_{k=1}^{N}|x_k|\right)^{p_N}\right\}$$
$$\leqslant\sup_N\left(\frac{1}{N}\sum_{k=1}^{N}|x_k-x_k^{(n)}|\right)^{p_N}\Bigg\}+\sup_N\left\{\left(\frac{1}{N}\sum_{k=1}^{N}|x_k^{(n)}|\right)^{p_N}\right\}$$
$$\leqslant\varepsilon+\sigma(p_N)(x^{(n)},0)\quad\text{for every } n\geqslant n_0$$

This means that $\{x_k\}\in Ces(p_n)_\infty$, so that $(Ces(p_n)_\infty, \sigma(p_n))$ is complete.

Next we show that the function $x\longrightarrow\lambda x$ is continuous for each fixed λ; for arbitrary $\varepsilon>0$, if

$$\sigma(p_n)(x,0)<\frac{\varepsilon}{\max\{1,|\lambda|\}},$$

then from the inequality

$$\sigma(\lambda x,\lambda y)<\max\{1,|\lambda|\}\sigma(x,y),$$

it follows that

$$\sigma(p_n)(\lambda x,0)<\max\{1,|\lambda|\}\sigma(p_n)(x,0)<\varepsilon.$$

This means that if λ is fixed, then the function $x\longrightarrow\lambda x$ is continuous at $x=0$, and hence continous at any x.

Finally we show that the function $(\lambda,x)\longrightarrow\lambda x$ is continuous at $(\lambda,x)=(0,0)$; for arbitrary $\varepsilon>0$, if $|\lambda|<1$ and $\sigma(p_n)(x,0)<\varepsilon$, then

$$\sigma(p_n)(x,0)\leqslant\max\{1,|\lambda|\}\sigma(p_n)(x,0)<\varepsilon.$$

This means that the function $(\lambda,x)\longrightarrow\lambda x$ is continuous at $(0,0)$.

REMARK 2.2. If $x=\{x_n\}\in Ces(p_n)_\infty$ and $\lambda\in\mathbf{C}$, then

$$\left(\frac{1}{n}\sum_{k=1}^{n}|\lambda x_k|\right)^{p_n}\leqslant\left(\frac{|\lambda|}{n}\sum_{k=1}^{n}|x_k|\right)^{p_n}$$
$$\leqslant\max\{1,|\lambda|\}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{p_n}.$$

Hence we have

$$\sup_n \left\{ \left(\frac{1}{n} \sum_{k=1}^{n} |\lambda x_k| \right)^{p_n} \right\} \leqslant \max\{1, |\lambda|\} \sup_n \left\{ \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{p_n} \right\} < \infty.$$

This shows that $\lambda x \in Ces(p_n)_\infty$. Therefore, it follows from Lemma 2.1 that $Ces(p_n)_\infty$ is a linear space over $\mathbf{C}$. However, for fixed x, the map $\lambda \longrightarrow \lambda x$ is not, in general, contiuuous at $\lambda=0$. Hence $(Ces(p_n)_\infty, v(p_n))$ is not, in general, a linear topological space.

PROPOSITION 2.3. *If* $0 < p_n \leqslant q_n \leqslant 1$ *for all* n, *then*

$$Ces(q_n)_\infty \subseteq Ces(p_n)_\infty.$$

Proof. If $\{x_n\} \in Ces(q_n)_\infty$, then there exists $M \geqslant 1$ such that

$$\left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{q_n} \leqslant M \text{ for all } n.$$

Hence it follows that

$$\left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{p_n} \leqslant M \text{ for all } n.$$

Therefore, $\{x_n\} \in Ces(p_n)_\infty$, and hence the result follows.

COROLLARY 2.4. *Suppose that* $0 < r_n, s_n \leqslant 1$ *for all* n, *and we write*

$$p_n = \min\{r_n, s_n\} \text{ and } q_n = \max\{r_n, s_n\}.$$

Then

(1) $Ces(q_n)_\infty = Ces(r_n)_\infty \cup Ces(s_n)_\infty$, *and*

(2) $Ces(p_n)_\infty = H$, *where* H *is the subspace of* S *generated by* $Ces(r_n)_\infty \cap Ces(s_n)_\infty$.

Proof. (1) It follows from Proposition 2.3 that $Ces(q_n)_\infty \subset Ces(r_n)_\infty$ and $Ces(q_n)_\infty \subset Ces(s_n)_\infty$. Hence we have $Ces(q_n)_\infty \subset Ces(r_n)_\infty \cap Ces(s_n)_\infty$.

Conversely, if $\{x_n\} \in Ces(r_n)_\infty \cap Ces(s_n)_\infty$, it follow from the inequality

$$|\lambda|^{q_n} \leqslant \max\{|\lambda|^{r_n}, |\lambda|^{s_n}\}$$

that for any n, we have

$$\left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{q_n} \leqslant \max\left\{ \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{r_n}, \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{s_n} \right\}$$
$$\leqslant \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{r_n} + \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{s_n}.$$

Thus it follows that

$$\sup_n \left\{ \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{q_n} \right\} \leqslant \sup_n \left\{ \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{r_n} \right\} + \sup_n \left\{ \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{s_n} \right\} < \infty.$$

Hence $\{x_n\}\in Ces(q_n)_\infty$, and therefore,

$$Ces(r_n)_\infty \cap Ces(s_n)_\infty \subset Ces(q_n)_\infty.$$

(2): Since $Ces(r_n)_\infty \subset Ces(p_n)_\infty$ and $Ces(s_n)_\infty \subset Ces(p_n)_\infty$, we have $Ces(r_n)_\infty \cup Ces(s_n)_\infty \subset Ces(p_n)_\infty$. Hence it follows that $H\subset Ces(p_n)_\infty$. To show the reverse inclusion, let

$$A=\{n: r_n\geqslant s_n\} \text{ and } B=\{n: r_n<s_n\}.$$

If $\{x_n\}\in Ces(p_n)_\infty$, we write

$$y_n=\begin{cases} x_n & \text{if } n\in A \\ 0 & \text{if } n\in B \end{cases} \quad \text{and} \quad z_n=\begin{cases} 0 & \text{if } n\in A \\ x_n & \text{if } n\in B \end{cases}$$

then we can show that

$$\sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^n|y_k|\right)^{s_n}\right\}\leqslant \sup_{n\in A}\left\{\left(\frac{1}{n}\sum_{k=1}^n|x_k|\right)^{p_n}\right\}<\infty,$$

$$\sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^n|z_k|\right)^{r_n}\right\}\leqslant \sup_{n\in B}\left\{\frac{1}{n}\sum_{k=1}^n|x_k|\right)^{p_n}\right\}<\infty$$

and hence

$$\{y_n\}\in Ces(s_n)_\infty\subset H, \quad \{z_n\}\in Ces(r_n)_\infty\subset H.$$

Therefore, $\{x_n\}=\{y_n\}+\{z_n\}\in H$. This proves that $Ces(p_n)_\infty\subset H$.

We write U for the set $\{x: x\in Ces(p_n)_\infty,\ \sigma(p_n)(x,0)\leqslant 1\}$. It is clear that $x\in U$ if and only if $\frac{1}{n}\sum_{k=1}^n|x_k|\leqslant 1$ for all n.

LLMMA 2.5. *If $0<p_n\leqslant q_n\leqslant 1$ for all n, then the set $Ces(q_n)_\infty$ is closed in $(Ces(p_n)_\infty, v(p_n))$. Consequently, $Ces(p_n)_\infty= Ces(q_n)_\infty$ if and only if $Ces(q_n)_\infty$ is dense in $(Ces(p_n)_\infty, v(p_n))$.*

Proof. Suppose $y\in Ces(q_n)_\infty$ and $x\in Ces(p_n)_\infty$-$Ces(q_n)_\infty$ Then it is clear that $x-y\notin Ces(q_n)_\infty\supset U$. Hence it follows that

$$\sigma(p_n)(x,y)=\sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^n|x_k-y_k|\right)^{p_n}\right\}>1.$$

This shows that x can not be a limit point of $Ces(q_n)_\infty$. Hence the result follows.

§3. Main results

In this section we give several equivalent conditions on $Ces(p_n)_\infty$ to be a topological linear space.

THEOREM 3. 1. *If* $0<p_n\leqslant q_n\leqslant 1$ *for all* n, *then the following three statements are equivalent*:

(1) $v(q_n)$ *is the topology induced on* $Ces(q_n)_\infty$ *by* $v(p_n)$.

(2) *The identity map* $(Ces(q_n)_\infty, v(q_n))\longrightarrow(Ces(q_n)_\infty, v(p_n))$ *is continuous.*

(3) *There exists* $P>1$ *such that* $Pp_n\geqslant q_n$ *for all* n.

Proof. (1) $\Longrightarrow$ (2): Since $v(q_n)$ is the topology induced on $Ces(q_n)_\infty$ by $v(p_n)$, it follows that every $G\in v(q_n)$ is of the form $G=Ces(q_n)_\infty\cap V$ for some $V\in v(p_n)$. This implies that the identity map is continuous.
(2) $\Longrightarrow$ (3): Suppose that (3) is not true; then there exists a sequence $n(1)<n(2)<\cdots$ such that $p_{n(N)}<\frac{1}{N}q_{n(N)}$ $(N=1, 2, \cdots)$.

We write

$$x^N=n(N)\cdot 2^{-\frac{1}{p_{n(N)}}}\cdot e_{n(N)}\quad (N=1, 2, \cdots).$$

Then we have

$$\sigma(p_n)(x^N, 0)=\left(\frac{1}{n(N)}\cdot n(N)\cdot 2^{-\frac{1}{p_{n(N)}}}\right)^{p_{n(N)}}=\frac{1}{2},$$

and

$$\sigma(q_n)(x^N, 0)=\left(\frac{1}{n(N)}\cdot n(N)\cdot 2^{-\frac{1}{p_{n(N)}}}\right)^{q_{n(N)}}$$
$$\leqslant(2^{-\frac{1}{p_{n(N)}}})^{Np_{n(N)}}=2^{-N}.$$

This shows that the identity map is not continuous at 0. Hence (3) follows.

(3) $\Longrightarrow$ (1): For every $\varepsilon>0$, let $\delta=\min\left\{1, \frac{\varepsilon}{2}\right\}$.

Then $\sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^n|x_k|\right)^{p_n}\right\}<\delta$ implies $\sup_n\left\{\frac{1}{n}\sum_{k=1}^n|x_k|\right)^{q_n}\right\}\leqslant\frac{\varepsilon}{2}<\varepsilon$.

This shows that

$$\{x\in Ces(q_n)_\infty : \sigma(p_n)(x, 0)<\delta\}$$
$$\subset\{x\in Ces(q_n)_\infty : \sigma(q_n)(x, 0)<\varepsilon\}.$$

This follows that $v(q_n)\subset v(p_n)\cap Ces(q_n)_\infty$.

To show the reverse inclusion, for given $\varepsilon>0$, let $\delta=\min\left\{1, \left(\frac{\varepsilon}{2}\right)^P\right\}$, then $\sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^n|y_k|\right)^{q_n}\right\}<\delta$ implies $\sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^n|y_k|\right)^{p_n}\right\}\leqslant\frac{\varepsilon}{2}<\varepsilon$.

This shows that

$$\{y\in Ces(q_n)_\infty : \sigma(q_n)(y,0)<\delta\}$$
$$\subset \{y\in Ces(q_n)_\infty : \sigma(p_n)(y,0)<\varepsilon\}.$$

Hence $v(p_n)\cap Ces(q_n)_\infty \subset v(q_n)$. Therefore it follows that the topologies they define are idendical. Hence (1) follows.

COROLLARY 3.2. *Let $\{p_n\}$ and $\{q_n\}$ be sequences of real numbers such that $p_n\leqslant q_n$ for all n. Then there exists $P>1$ such that $Pp_n\geqslant q_n$ for all n if and only if $(Ces(p_n)_\infty, v(p_n))=(Ces(q_n)_\infty, v(q_n))$.*

Proof. ($\Longrightarrow$): If $x\in Ces(p_n)_\infty$ then there exists $M\geqslant 1$ such that

$$\sup_n\left\{\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{p_n}\right\}\leqslant M.$$

This follows that

$$\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{p_n}\leqslant M \text{ for all } n$$

and hence

$$\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{Pp_n}\leqslant M^P \text{ for all } n.$$

Therefore, $\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{q_n}\leqslant M^P$ for all n, so that $x\in Ces(q_n)_\infty$. This shows that $Ces(p_n)_\infty\subset Ces(q_n)_\infty$. Hence it follows from Proposition 2.3, and Theorem 3.1 ((3) implies (1)) that $(Ces(p_n)_\infty\ v(p_n))=(Ces(q_n)_\infty, v(q_n))$.

($\Longleftarrow$): The proof follows from Theorem 3.1 ((1) implies (3)).

THEOREM 3.3 *The following four statements on $\{p_n\}$ are equivalent:*

(1) *$v(\infty)$ is the topology induced on Ces (∞) by $v(p_n)$.*

(2) *The identity map $(Ces(\infty), v(\infty))\longrightarrow(Ces(\infty), v(p_n))$ is continuous.*

(3) $\inf p_n>0$.

(4) *$(Ces(p_n)_\infty, v(p_n))$ is a linear topological space.*

Proof. The equivalence of (1)$\sim$(3) follows from Theorem 3.1 with $q_n=1$ for all n.

(3)$\Longrightarrow$(4): If $\inf p_n=p>0$, then we have

$$\sigma(\lambda x, 0)\leqslant \max\{|\lambda|, |\lambda|^p\}\sigma(x,0).$$

For every $\varepsilon>0$, let δ such that $0<\delta<\min\left\{1, \left(\frac{\varepsilon}{\sigma(x,\theta)}\right)^{\frac{1}{p}}\right\}$. Then $|\lambda|<\delta$ implies

$$\begin{aligned}\sigma(\lambda x,\ 0) &\leqslant \max\{|\lambda|, |\lambda|^p\}\,\sigma(x,0)\\ &= |\lambda|^p\sigma(x,0) < \delta^p\sigma(x,0)\\ &< \left(\left(\frac{\varepsilon}{\sigma(x,0)}\right)^{\frac{1}{p}}\right)^p\sigma(x,0)=\varepsilon.\end{aligned}$$

This shows that the function $\lambda \longrightarrow \lambda x$ is continuous at $\lambda=0$ for every fixed x, and hence continuous everywhere. It follows from Lemma 2.1 and Remark 2.2 that $(Ces(p_n)_\infty, v(p_n))$ is a linear topological space.

(4) $\Longrightarrow$ (3): If (3) is not true, then $p_n=0$ and $x_n=1$ for all n. Hence $\sigma(\lambda x, 0)=1$ for all λ with $0<|\lambda|\leqslant 1$. Thus $\lambda x \not\longrightarrow 0$ as $\lambda \longrightarrow 0$. This shows that $(Ces(p_n)_\infty, v(p_n))$ is not a linear topological space. Hence (4) implies (3).

References

1. C. M. Leibowitz, *A note on the Cesàro Sequence Spaces*, Tamkang J. of Math. 2 (1971), 151-157.
2. Jau-shyong Shiue, *On the Cesàro Sequence Spaces*, Tamkang J. of Math. **1**(1970), 19-25.
3. J. L. Kelley, I. Namioka, *Linear Topological Spaces*, D. Van Nostrand Co., New York, 1963.
4. Kwang Ho Shon, *On the Cesàro Sequence Spaces*, Ulsan Institute of Technology Report **11**(1980).
5. R. B. Ash, *Real Analysis and Probability*, Academic Press, New York, 1972.
6. S. Simons, *The Sequence Spaces $l(p_v)$ and $M(p_v)$*, Proc. London Math. Soc. (3) **15**(1965), 426-436.

Ulsan Institute of Technology